On invariance properties of empirical laws*

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Abstract

The concept of invariance has played a central role in the consideration of candidate functions for empirical laws. Such laws typically are assumed invariant under allowable changes in representation of the variables, that is, under certain strictly increasing, surjective transformations that act individually on the variables. The nature of the invariance and the specification of which transformations are allowable may be open to interpretation. In this paper, we present two possible interpretations of invariance, in the spirit of Falmagne and Narens (1983), and we examine the relationship between the two. Our main result is a generalization of a theorem by Falmagne and Narens (1983) which gives a condition under which the two interpretations are equivalent. The generalization was motivated by the observation that there are important cases in which invariance holds under transformations that cannot be written as functions on separate variables. Our results aim at a revised approach to the consideration of candidate empirical laws, one which allows broader notions of invariance to better classify actual scientific laws, some of which may not satisfy certain invariances.

Notions of invariance have played a central role in the investigation of statements considered suitable to be scientific laws. For instance, the classical concept of 'dimensional invariance' has been widely used, via the method of dimensional analysis, in the search for lawful numerical relations among physical variables. The method of dimensional analysis may be employed, for example, in the derivation of the functional description of the motion of a simple pendulum (see e.g. Krantz et al., 1971; Narens, 2002). A related invariance notion, 'meaningfulness', has been used in the theoretical sciences for seemingly the same purpose as dimensional analysis: scientists seek to describe empirical relationships among variables via functional laws, and putative invariances of the measurement theories of these variables may greatly constrain the possible forms of such laws. The specific use of these and related notions of invariance in the formulation of lawful functional relations may be found, for example, in Luce (1959, 1964); Osborne (1970); Falmagne and Narens (1983); Aczel et al. (1986); Kim (1990).

The focus of the present paper is a comparison of these two notions of invariance, which are appropriately formalized here in the spirit of Falmagne and Narens (1983). Our main result, which gives insight into the relationship between the two notions, generalizes a result by these authors. In preparation for a formal presentation of the two notions and the properties which connect them, we examine an introductory example.

The pressure (P), volume (v), temperature (t) and quantity (n) of an "ideal" gas are related by the equation

$$P(v,t,n) = R\frac{1}{v}nt, \tag{1}$$

in which R is a dimensional constant. Note that the numerical value of R depends on the units employed in the measurement of the variables. Let us fix some triple of units in Eq. (1), say, liters, degrees Kelvin, and moles. Any change of units for one of the variables amounts to multiplication of one of these fixed units by a positive number. Suppose we change to a triple of units whose volume measure requires multiplication of liters by α, whose temperature measure requires multiplication of degrees Kelvin by β, and whose quantity measure requires multiplication of moles by γ. Defining the functions $f_1, f_2, f_3 : \mathbb{R}^+ \to \mathbb{R}^+$ by $f_1(x) = \alpha x$, $f_2(x) = \beta x$, and $f_3(x) = \gamma x$, and setting $f = (f_1, f_2, f_3)$, it is appropriate to write the equation relating the variables as

$$P_f(v,t,n) = R(f)\frac{1}{v}nt, \tag{2}$$

indicating the particular dependence on the units employed. The functions f_1, f_2, and f_3 are called 'representations', with each amounting essentially to a choice of unit for a particular variable. Note that, with this notation, Eq. (1) would be rewritten $P_{\tilde{\iota}}(v,t,n) = R(\tilde{\iota})\frac{1}{v}nt$, where $\tilde{\iota} = (\iota,\iota,\iota)$ for ι the identity function (defined by $\iota(x) = x$) on $\mathbb{R}^+$.

A minimal requirement for a law relating physical variables is that the particular choice of representations should not alter the numerical description of the phenomenon in any essential manner. This intuitive notion may be subject to different interpretations; we propose one of them here. Suppose we measure the pressures of an ideal gas at two different triples of volume, temperature, and quantity, using the representations f_1, f_2, and f_3, and we find that the first pressure is less than or equal to the second. The relationship between the two pressure computations should hold even if we use different representations g_1, g_2, and g_3. In other words, it should be the case that, for any representations g_1, g_2, g_3, with $g = (g_1, g_2, g_3)$, we have

$$P_f(f(v,t,n)) \leq P_f(f(v',t',n'))$$

(3) $$\text{iff}$$

$$P_g(g(v,t,n)) \leq P_g(g(v',t',n')).$$

Note that the function P in Eq. (2) satisfies this requirement. Indeed, with f_1, f_2, and f_3 as above, we have

$$P_f(f(v,t,n)) \leq P_f(f(v',t',n'))$$

$$\text{iff}$$

$$R(f)\frac{1}{\alpha v}\gamma n\beta t \leq R(f)\frac{1}{\alpha v'}\gamma n'\beta t'$$

$$\text{iff}$$

$$\frac{1}{v}nt \leq \frac{1}{v'}n't'.$$

As this last equality does not depend on the representations used, Formula (3) follows for any functions f and g specifying the representations. We shall say that the function P satisfies the property of 'meaningfulness'. (A precise definition is given as Definition 5.)

Meaningfulness has been described as follows:

> A numerical statement is meaningful if and only if its truth (or falsity) is constant under admissible scale transformations of any of its numerical assignments, that is, any of its numerical functions expressing the results of measurement. (Suppes and Zinnes, 1963, p. 66)

(Here, "scale" corresponds to "representation.") This description of meaningfulness is ambiguous and may lead to more than one mathematical formulation. The equivalence in (3) provides one such formulation: constancy of the truth of a statement is interpreted as preservation of the order of functional outputs, and admissible transformations are those which match the transformations on which the functions depend. There may be other interpretations of this description of meaningfulness, however. For instance, consider a fixed P_f, and suppose that there are triples (v,t,n) and (v',t',n') such that

$$P_f(v,t,n) \leq P_f(v',t',n').$$

If for any representations g_1, g_2, and g_3, with $g = (g_1, g_2, g_3)$, we have

$$P_f(g(v,t,n)) \leq P_f(g(v',t',n'))$$

$$\text{iff}$$

$$P_f(v,t,n) \leq P_f(v',t',n'),$$

then P_f satisfies an invariance property which may be said to satisfy the above description of meaningfulness. However, we shall say in this case that P_f is 'dimensionally invariant'. A formal definition of dimensional invariance is given as Definition 6 (see also Causey, 1969; Krantz et al., 1971; Narens, 2002).

The notions of meaningfulness and dimensional invariance are thus seen to be closely related. The two may be hard to separate; indeed, it may seem that any empirical relation that satisfies one must satisfy the other. We will see through the following example that this is not the case.

1. Example. Choose representations f_1 and f_2 of length and (positive) temperature difference, respectively, and write $f = (f_1, f_2)$. The final length L of a rod of initial length ℓ following an increase t in temperature is given by the equation

$$L_f(\ell, t) = \ell(1 + \zeta(f_2)\, t),$$

in which ζ is a constant that depends on f_2. In particular, if f_2 is the representation corresponding to multiplication by β, then $\zeta(f_2) = \frac{\zeta(\iota)}{\beta}$, where again ι is the identity function on $\mathbb{R}^+$. Then the function L_f satisfies meaningfulness but not dimensional invariance. (This will be demonstrated below in the Definitions and Basic Concepts section.)

We present a result in Theorem 11—the main result of this paper—which ties together the notions of meaningfulness and dimensional invariance. In particular, we show that, under a natural condition relating members of a family of functions, the two notions are equivalent.

As mentioned, our main result is a generalization of a result by Falmagne and Narens (1983). The generalization was motivated by close inspection of the types of transformations under which invariance may be studied. Note that each of the transformations considered so far is made up of individual transformations which act independently on separate variables. For instance, the transformation f considered in Example 1 is written $f = (f_1, f_2)$ for the two transformations f_1 and f_2, each of which acts on a single variable. Such transformations are the ones typically considered in the measurement literature (see Narens, 2002). There are important situations, though, in which significant invariances hold under transformations that can not be written as individual transformations on separate variables. For instance, consider the transformation of $\triangle ABC$ to $\triangle ABD$ as shown in Figure 1, in which segment BD is constructed parallel to AC, with the length of BD equal to that of AC. Clearly the area of the triangle is invariant under this transformation. If we define the transformation via the function $f : \prod_{i=1}^{3}]1, 2[\to \mathbb{R}^+$, where $f(a, b, c) = (a, d, e)$, then there are no functions f_1, f_2, and f_3 such that $f = (f_1, f_2, f_3)$. In other words, f is not 'factorizable.' (See Definition 4 below.)

We give two more examples of transformations which are not factorizable, but under which important invariances hold.

2. Example. Psychophysicists are interested in the relationships between physical magnitudes of stimuli and the strengths of the sensations they evoke (Fechner, 1860). An important task in psychophysics is the construction of a measure of 'subjective distance' between stimuli based on data which give, for instance, the probability that one stimulus

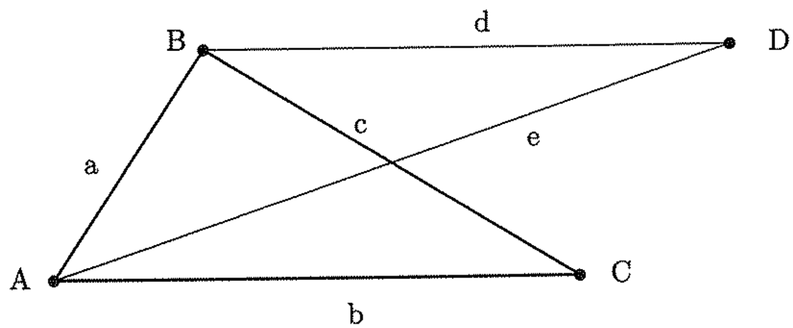

Figure 1: Depiction of a transformation that is not factorizable

is judged to be different from another. This task, referred to as *Fechnerian scaling*, may be complicated by the fact that the relevant stimuli occupy a multidimensional space. For instance, the stimuli might be auditory tones that vary in both amplitude and frequency. Dzhafarov and Colonius (2001) propose a theory of Fechnerian scaling which is built in part upon the idea that such distance measures must be invariant with respect to any diffeomorphic transformation of the space of stimulus magnitudes (usually taken to be a subset of $\mathbb{R}^n$). Obviously, such transformations may not be factorizable in the multidimensional case.

3. Example. In the theory of relativity, the "form" of a physical law must be invariant under a particular transformation of the variables called the Lorentz transformation:

> Every general law of nature must be so constituted that it is transformed into a law of exactly the same form when, instead of the space-time variables x, y, z, and t of the original co-ordinate system K, we introduce new space-time variables x', y', z', t' of a co-ordinate system K'. In this connection the relation between the ordinary and the accented magnitudes is given by the Lorentz transformation. Or in brief: General laws of nature are co-variant with respect to Lorentz transformations. (Einstein, 1961, pp. 42-43).

This transformation is given by

$$(x, y, z, t) \mapsto (x', y', z', t') = \left(\frac{x - \nu t}{\sqrt{1 - (\frac{\nu}{c})^2}}, y, z, \frac{t - \frac{\nu}{c^2} x}{\sqrt{1 - (\frac{\nu}{c})^2}} \right),$$

where x, y, and z are position coordinates, t is time, c is the speed of light, and ν is the velocity of coordinate system K' with respect to K (in the direction of the x-axis of K). It is clear that the transformation is not factorizable.

Definitions and Basic Concepts

Let X be a nonempty set, and let $\mathcal{F} = \{f | f : X \overset{\text{onto}}{\rightarrow} X\}$ be a family of surjective functions mapping X onto itself. For any $f \in \mathcal{F}$, let M_f be a function mapping X to a linearly ordered set Z, with the order written $(Z, \leq)$. In the examples above, $X \subseteq \mathbb{R}^n$ and $Z \subseteq \mathbb{R}$.

We call $\mathcal{M} = \{M_f \,|\, f \in \mathcal{F}\}$ a *family of ordinal codes*. Each $M_f \in \mathcal{M}$ is an *ordinal code*.

In this section, we present formally the concepts of meaningfulness and dimensional invariance. We emphasize that the transformations involved may or may not be factorizable. The precise definition of factorizability is as follows:

4. Definition. Suppose $X = \prod_{i=1}^{n} X_i$ for nonempty sets $X_1, \ldots, X_n$. A function $f : X \rightarrow Z$ is *factorizable* if there exist functions $f_i : X_i \rightarrow Z$, for $i = 1, \ldots, n$, such that $f(x_1, \ldots, n) = (f_1(x_1), \ldots, f_n(x_n))$ for all $(x_1, \ldots, x_n) \in X$.

The following two definitions formalize and generalize the concepts of meaningfulness and dimensional invariance introduced earlier through examples.

5. Definition. The ordinal code $M_f \in \mathcal{M}$ is *meaningful* if, whenever $f^* \in \mathcal{F}$, we have

$$M_f[f(x)] \leq M_f[f(y)]$$

iff

$$M_{f^*}[f^*(x)] \leq M_{f^*}[f^*(y)]$$

for all $x, y \in X$. If this holds for all $M_f \in \mathcal{M}$, we say that $\mathcal{M}$ is *meaningful.*

6. Definition. The ordinal code $M_f \in \mathcal{M}$ is *dimensionally invariant* if, whenever $f^*, g^* \in \mathcal{F}$, we have

$$M_f[f^*(x)] \leq M_f[f^*(y)]$$

iff

$$M_f[g^*(x)] \leq M_f[g^*(y)]$$

for all $x, y \in X$. If this holds for all $M_f \in \mathcal{M}$, we say that $\mathcal{M}$ is *dimensionally invariant.*

As mentioned, though the notions of meaningfulness and dimensional invariance are related, there exist physical laws which satisfy one but not the other. We return to Example 1, which presents a law that is meaningful but not dimensionally invariant.

Example 1 revisited. Choose representations f_1 and f_2 of length and (positive) temperature difference, respectively, and write $f = (f_1, f_2)$. The final length L of a rod of initial length ℓ following an increase t in temperature is given by the equation

$$L_f(\ell, t) = \ell(1 + \zeta(f_2)\, t), \tag{4}$$

in which ζ is a constant that depends on f_2. In particular, if f_2 is the representation corresponding to multiplication by β, then $\zeta(f_2) = \frac{\zeta(\iota)}{\beta}$.

To see that meaningfulness is satisfied, suppose that the representations f_1 and f_2 correspond to multiplication by α and β, respectively. Then

$$L_f(f(\ell, t)) \leq L_f(f(\ell', t'))$$

iff

$$\alpha\ell(1+\zeta(f_2)\,\beta t) \le \alpha\ell'(1+\zeta(f_2)\,\beta t')$$

iff

$$\alpha\ell(1+\frac{\zeta(\iota)}{\beta}\,\beta t) \le \alpha\ell'(1+\frac{\zeta(\iota)}{\beta}\,\beta t')$$

iff

$$\ell(1+\zeta(\iota)\,t) \le \ell'(1+\zeta(\iota)\,t'),$$

and this final inequality does not depend on the representations f_1 and f_2. Thus, L_f is meaningful, as an equivalence similar to the one in (3) is satisfied. Now we show that L_f is not dimensionally invariant. We let f_1 correspond to multiplication by 1, f_2 correspond to multiplication by $\zeta(\iota)$, g_1 correspond to multiplication by 1, and g_2 correspond to multiplication by 2. Setting $\ell = 1$, $\ell' = 2$, $t = 3$, $t' = 1$, and $g = (g_1, g_2)$, we have

$$L_f(\ell,t) = 1(1+3) = 4 \le 2(1+1) = L_f(\ell',t'),$$

but

$$L_f(g(\ell,t)) = 1(1+(2)3) = 7 > 2(1+(2)1) = L_f(g(\ell',t')).$$

This means that L_f is not dimensionally invariant. We note that there actually are several physical laws having the form in Eq. (4), including Guy Lussac's Law (for the change in volume of an ideal gas under a temperature change) and the Lorentz contraction (for the change in length of a rod under a velocity change); see e.g. Hix and Alley (1958).

It turns out that the notions of meaningfulness and dimensional invariance are independent: in addition to the function above, which is meaningful but not dimensionally invariant, there exist functions which are dimensionally invariant but not meaningful. As an example, consider the function $M_f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ defined by $M_f(x,y) = x + \lambda y$, where $f = (f_1, f_1)$ and f_1 corresponds to multiplication by λ. As shown in Falmagne and Narens (1983) and Roberts (1985), this function is not meaningful, but it is dimensionally invariant. In contrast to Example 1, this and other available examples of functions which are not meaningful are hypothetical, i.e., are not necessarily associated to any extant empirical laws. This is not surprising, in view of the compelling argument behind our formulation of meaningfulness. Indeed, this argument probably has been a part of scientists' intuition since long before a definition was formalized. Moreover, though empirical relations may have been introduced that did not satisfy this notion, these relations probably were eliminated.

The following Lemma is a well-known result, so the proof is omitted. (See, e.g., Munkres (1975).)

7. Lemma. *Suppose $f : X \to Y$, with X and Y ordered sets in the order topology. If f is strictly increasing and surjective, then f is a homeomorphism (i.e., a bicontinuous bijection).*

The next two Propositions are of use in the proof of Theorem 11.

8. Proposition. *The family $\mathcal{M}$ is meaningful if, and only if, for each $f, h \in \mathcal{F}$ there exists a strictly increasing $H_{f,h} : M_h(X) \to M_f(X)$ such that*

$$H_{f,h}(M_h[h(x)]) = M_f[f(x)]$$

for all $x \in X$. In this case, $H_{f,h}$ also is surjective.

Moreover, if Z has the order topology, then $H_{f,h}$ and $H_{f,h}^{-1}$ are continuous.

PROOF. Choose $f, h \in \mathcal{F}$.

($\Rightarrow$): Suppose $\mathcal{M}$ is meaningful. Define the function $H_{f,h}$ by

$$H_{f,h}(M_h[h(x)]) = M_f[f(x)]$$

for all $x \in X$. Then $H_{f,h}$ is well defined and strictly increasing since $\mathcal{M}$ is meaningful. Also, since f and h map X onto itself, $H_{f,h}$ maps $M_h(X)$ onto $M_f(X)$.

($\Leftarrow$): Let $f^* \in \mathcal{F}$. Suppose $H_{f,h}$ and $H_{f^*,h}$ are as described in the statement of the proposition. Since $H_{f,h}$ and $H_{f^*,h}$ are strictly increasing, we have for all $x, y \in X$,

$$M_f[f(x)] = H_{f,h}(M_h[h(x)]) \leq H_{f,h}(M_h[h(y)]) = M_f[f(y)]$$

iff

$$M_h[h(x)] \leq M_h[h(y)]$$

iff

$$M_{f^*}[f*(x)] = H_{f^*,h}(M_h[h(x)]) \leq H_{f^*,h}(M_h[h(y)] = M_{f^*}[f*(y)].$$

Therefore, we have $M_f[f(x)] \leq M_f[f(y)] \Leftrightarrow M_{f^*}[f^*(x)] \leq M_{f^*}[f^*(y)]$, so $\mathcal{M}$ is meaningful.

The *Moreover* statement is proved with an application of Lemma 7. □

9. Proposition. *The family $\mathcal{M}$ is dimensionally invariant if, and only if, for each $M_f \in \mathcal{M}$ and for all $g, g^* \in \mathcal{F}$ there exists a strictly increasing $Q_{f,g,g^*} : M_f(X) \to M_f(X)$ such that*

$$Q_{f,g,g^*}(M_f[g^*(x)]) = M_f[g(x)]$$

for all $x \in X$. In this case, Q_{f,g,g^*} also is surjective.

Moreover, if Z has the order topology, then Q_{f,g,g^*} and Q_{f,g,g^*}^{-1} are continuous.

We omit the proof, which is similar to that of Proposition 8.

The following definition gives a property which provides a link between the two notions of invariance. This property applies to families of ordinal codes, and it requires that any two codes be related by in a natural way, that is, via a mapping that depends only on the indexing transformations.

10. Definition. The family of ordinal codes $\mathcal{M}$ is *isotone* if there exists a function $M^* : X \to Z$ such that, for each $M_f \in \mathcal{M}$, we have $M_f = m_f \circ M^*$ for some strictly increasing and surjective $m_f : M^*(X) \to M_f(X)$.

Note that there is no loss of generality in assuming that $M^* = M_h$ for any $h \in \mathcal{F}$. Indeed, if $\mathcal{M}$ is isotone, then $M_h = m_h \circ M^*$ and $M_f = m_f \circ M^*$ for functions M^*, m_h, and m_f as in Definition 10, with $f \in \mathcal{F}$. But then $M_f = (m_f \circ m_h^{-1}) \circ M_h$, and $m_f \circ m_h^{-1} : M_h(X) \to M_f(X)$ is strictly increasing and surjective.

Main Result

The following theorem, which generalizes Theorem 4 in Falmagne and Narens (1983), specifies the relationship among meaningfulness, dimensional invariance, and isotonicity. In particular, it states that meaningfulness and dimensional invariance are equivalent for isotone families of ordinal codes.

11. Theorem. *Any two of the properties of meaningfulness, dimensional invariance, and isotonicity imply the third.*

PROOF.
(i) Dimensional invariance and isotonicity imply meaningfulness:
Choose $g^* \in \mathcal{F}$. For any $f \in \mathcal{F}$ and $x \in X$, we have

$$\begin{aligned} M_f[f(x)] &= Q_{f,f,g^*}(M_f[g^*(x)]) && \text{[by Prop. 9]} \\ &= (Q_{f,f,g^*} \circ m_{f,g^*})(M_{g^*}[g^*(x)]) && \text{[by isotonicity].} \end{aligned}$$

Since $Q_{f,f,g^*} \circ m_{f,g^*}$ is strictly increasing, Prop. 8 gives that $\mathcal{M}$ is meaningful.

(ii) Meaningfulness and isotonicity imply dimensional invariance:
Suppose $\mathcal{M}$ is meaningful and isotone, and let $M_f, M_h \in \mathcal{M}$. Since $\mathcal{M}$ is meaningful, there exists a strictly increasing $H_{f,h}$ such that

$$M_f[f(x)] = H_{f,h}(M_h[h(x)])$$

for all $x \in X$. Since $\mathcal{M}$ is isotone, there exists a strictly increasing and surjective m_f such that

$$M_f[f(x)] = m_{f,h}(M_h[f(x)])$$

for all $x \in X$. Thus,

$$M_h[f(x)] = m_{f,h}^{-1}(M_f[f(x)]) = (m_{f,h}^{-1} \circ H_{f,h})(M_h[h(x)]), \tag{5}$$

where $m_{f,h}^{-1} \circ H_{f,h}$ is strictly increasing.

Let $g \in \mathcal{F}$. We have

$$\begin{aligned} M_g[f(x)] &= m_{g,h}(M_h[f(x)]) && \text{[by isotonicity]} \\ &= (m_{g,h} \circ m_{f,h}^{-1} \circ H_{f,h})(M_h[h(x)]) && \text{[by Eq. (5)]} \\ &= (m_{g,h} \circ m_{f,h}^{-1} \circ H_{f,h} \circ m_{h,g}^{-1})(M_g[h(x)]) && \text{[by isotonicity],} \end{aligned}$$

where $m_{h,g} \circ m_f^{-1} \circ H_{f,h} \circ m_{h,g}^{-1}$ is strictly increasing. Therefore, by Prop. 9, M_g is dimensionally invariant. Since $g \in \mathcal{F}$ is arbitrary, we have that $\mathcal{M}$ is dimensionally invariant.

(iii) Dimensional invariance and meaningfulness imply isotonicity:
Suppose $\mathcal{M}$ is dimensionally invariant and meaningful, and choose $M_h \in \mathcal{M}$. Let $f \in \mathcal{F}$ be arbitrary.

Since $\mathcal{M}$ is meaningful, there exists a strictly increasing and surjective $H_{f,h}$ such that

$$M_f[f(x)] = H_{f,h}(M_h[h(x)])$$

for all $x \in X$.

Since $\mathcal{M}$ is dimensionally invariant, there exists a strictly increasing and surjective $Q_{h,h,f}$ such that

$$M_h[h(x)] = Q_{h,h,f}(M_h[f(x)])$$

for all $x \in X$.

Thus,

$$M_f[f(x)] = (H_{f,h} \circ Q_{h,h,f})(M_h[f(x)])$$

for all $x \in X$, where $H_{f,h} \circ Q_{h,h,f} : M_h(X) \to M_f(X)$ is strictly increasing and surjective. Since $f : X \to X$ is surjective, we have for all $a \in X$ that

$$M_f[a] = (H_{f,h} \circ Q_{h,h,f})(M_h[a]),$$

i.e., $\mathcal{M}$ is isotone. □

Discussion

We have compared the notions of dimensional invariance and meaningfulness in the context of arbitrary transformations on the set of inputs. The results in Theorem 11 generalize those of Falmagne and Narens (1983), who consider invariance only under transformations which can be factorized and written as strictly increasing, surjective, real-valued functions of real variables. These results state that dimensional invariance and meaningfulness are equivalent for families of functions whose members are related via strictly increasing functions.

We have shown, through Example 1, that dimensional invariance and meaningfulness are distinct among extant physical laws, that is, there exist physical laws which satisfy one condition of invariance but not the other. For instance, the law given in Example 1 is meaningful but not dimensionally invariant. However, note that this law may naturally be rewritten

$$\Delta L_f(\ell, t) = \ell\, \zeta(f_2)\, t, \tag{6}$$

where $\Delta L = L(\ell, t) - \ell$, often the quantity of interest. It is straightforward to show that ΔL_f in Eq. (6) is both meaningful and dimensionally invariant. (In fact, under certain assumptions of differentiability, the transformation $\phi(\ell) = \ell$ is the only transformation that renders $L(\ell, t) - \phi(\ell)$ meaningful and dimensionally invariant.) One wonders whether dimensional invariance may be unessential: perhaps a law may always be trivially rewritten in a way that recovers dimensional invariance. This does not appear to be the case, as demonstrated by the following:

$$P_f(s, t) = \frac{1}{1 + e^{\frac{s - \xi(f)}{\kappa(f)\, t}}}, \tag{7}$$

where ξ and κ are constants which may depend on the representations f_1 and f_2 of $f = (f_1, f_2)$. (Equation (7) gives the probability $P_f(s, t)$ that an electron will exist at an energy state s at absolute temperature t. The constant κ is Boltzmann's constant, and ξ is the Fermi level energy.) Examinations of physical laws which are not dimensionally invariant, of whether these laws allow associated formulations which are dimensionally invariant, and of how those associated formulations are obtained are lines of future research. These lines suggest the use of dimensional invariance beyond the typical use in classical physics, i.e., beyond the method of dimensional analysis.

Putative "laws" which are invariant under the Lorentz transformation are particularly interesting because they may be studied both with respect to this transformation and with respect to changes of representation. It it feasible that some of these may not be invariant under changes in representation, or at least would not satisfy dimensional invariance in the sense of Definition 6, when only the changes of representation are considered. Note that a study of such "laws" necessarily involves an approach in which invariance notions (i) are stated with suitable generality for the transformations, and (ii) have families of functions as the objects of interest, rather than single functions, as is the approach typically taken (cf. Causey, 1969; Krantz et al., 1971; Luce, 1978; Narens, 2002). The formulations in the present paper are appropriate for such a study.

The motivation for this study, and perhaps for any study of properties of invariance, is the investigation of the role of invariance in limiting the possible forms that a scientific law may take. As mentioned, there is a literature which seeks to pinpoint the functional forms which may relate independent and dependent variables that are allowed certain types of representations (e.g. Luce, 1959, 1964; Osborne, 1970; Falmagne and Narens, 1983; Aczel et al., 1986; Kim, 1990). These functions are assumed to satisfy certain invariance properties, and quite often these properties are analogous to the notion of classical dimensional invariance (Luce, 1959, 1964; Osborne, 1970; Aczel et al., 1986). (We specify "classical" because the invariance is assumed for a single function, rather than for a family of functions as in the present paper and in Falmagne and Narens (1983).) Considering the laws given by Equations (4) and (7)—established laws which do not satisfy dimensional invariance—it may be necessary to revise the assumptions of invariance in these investigations. Invariances which lead to the derivation of functional forms that include (4) and (7) must necessarily be weaker than dimensional invariance, but it is not obvious how to proceed to derive forms as diverse as these two functions. Perhaps a fruitful approach is to consider families of functions, only some of whose members satisfy an invariance property. This approach, along with a better understanding of manipulations which may recover dimensional invariance for functions (and families) that are not dimensionally invariant, could lead to more appropriate categorizing of functional forms for scientific laws.

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