Generic absoluteness and the Continuum Problem

John R. Steel

University of California, Berkeley

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A role for Philosophy?

1. Philosophical considerations may have a role to play in an eventual solution to the Continuum Problem, since any solution will probably need to be accompanied by some analysis of *what it is to be a solution*.

2. Conversely, the Continuum Problem presents philosophers with an important case study.

3. The "official" Cabal philosophy has been dubbed *consciously naive realism*. This was an appropriate attitude when the founding fathers were first laying down the new large cardinals/determinacy theory. One had the axioms; the important thing was to develop them.

4. It may be useful now to attempt a more sophisticated realism, one accompanied by some self-conscious, metamathematical considerations related to meaning and evidence in mathematics.

Maximize

1. All mathematical language can be translated into the language of set theory, and all "ordinary" theorems proved in ZFC.
2. In extending ZFC, we are attempting to *maximize interpretative power*.
3. To adopt "there are measurable cardinals" is to seek to naturally interpret all mathematical theories of sets, to the extent they have natural interpretations, in extensions of ZFC + "there is a measurable cardinal".
4. Maximizing interpretative power entails maximizing consistency strength (but not conversely).

V=L vs. Maximize

V=L is restrictive, in that adopting it limits the interpretative power of our language.

1. The language of set theory as used by the V=L believer can be translated into the language of set theory as used by the "∃ measurable cardinal" believer:

$$\varphi \mapsto \varphi^L .$$

2. There is no translation in the other direction.

3. Proving φ in ZFC + V=L is equivalent to proving φ^L in ZFC. Thus adding V=L *settles no questions* not already settled by ZFC. It just prevents us from asking as many questions!

4. The Foundation Axiom, V=WF, is not similarly restrictive, because we know of no interesting structure outside WF. In the case of L, we have $0^\sharp$, etc.

The Instrumentalist Dodge

Given a theory T and class of sentences Γ, Inst(T, Γ) is the theory: All theorems of T in Γ are true.

Thus

$$\text{Inst}(T, \Pi^0_1) \equiv \text{Con}(T),$$

$$\text{Inst}(T, \Pi^0_2) \equiv 1 - \text{Con}(T).$$

One could obtain all the Σ^1_2 consequences of measurables in

$$\text{ZFC + V=L + Inst(There are measurables , } \Sigma^1_2).$$

One could obtain all the Π^0_1 consequences of measurables in

$$\text{PA + Inst(There are measurables , } \Pi^0_1).$$

There are endless variations here. They are all parallel to the theory: “There are no electrons, but mid-size objects behave as if there were.”

In general, a retreat from T to Inst(T, Γ) has no practical significance, unless one has a proposal for some better instrument S which is incompatible with T in the realm of non-Γ statements.

V=L versus V=L[G]

Some foundational conflicts are not real. For example, set theory with proper classes, vs. set theory without proper classes. Another example is probably provided by Aczel set theory versus ZFC. In both situations we have different, but intertranslatable, ways of using the syntax of the language of set theory.

Thought experiment: *A* and *B* accept ZFC and all the large cardinals consistent with V=L, but reject $0^\sharp$. In fact, *A* believes V=L, while *B* believes "V=L[G], where *G* is generic over L for the poset which adds ω_2 Cohen reals. *A* and *B* seem to disagree on CH.Is there a real conflict here?

Maximize does not decide between them, since *B* can interpret *A*'s φ as φ^L, and *A* can interpret *B*'s φ as $\emptyset \Vdash \varphi$.

In fact, if there is indeed a conlict, it has no practical significance, since developing one point of view is equivalent to developing the other.

The one true hierarchy

We have found natural new axioms, the large cardinal axioms. These are plausible strengthenings of the axiom of infinity of ZFC.

1. These axioms have proved crucial to organizing and understanding the family of possible extensions of ZFC.
   (a) Many natural extensions T of ZFC have been shown to be consistent relative to some large cardinal hypothesis H, via the method of forcing.
   (b) Often, it has been shown that the consistency of the large cardinal hypothesis H must be assumed, in that Con(T) implies Con(H). This involves a canonical inner model for H. Such a construction also provides strong evidence that H is indeed consistent.

Examples

- Con(ZF + All sets Lebesgue measurable)

  $\leftrightarrow$ Con(ZFC + There is an inaccessible),

- $$\text{Con}\left(\text{ZFC} + \begin{array}{l}\text{There is a total extension}\\ \text{of Lebesgue measure}\end{array}\right)$$

  $\leftrightarrow$ Con(ZFC + There is a measurable),

- Con(ZFC+ GCH first fails at $\aleph_\omega$)

  $$\leftrightarrow \left(\text{ConZFC} + \begin{array}{l}\text{There is a measurable}\\ \kappa \text{ of order } \kappa^{++}\end{array}\right)$$

- Con(ZFC + All games in $L(\mathbb{R})$ are determined)

  $$\leftrightarrow \text{Con}\left(\text{ZFC} + \begin{array}{l}\text{There are infinitely}\\ \text{many Woodin cardinals}\end{array}\right)$$

- Con(ZFC + There is a supercompact cardinal)

  $\rightarrow$ Con(ZFC + Proper forcing axiom)

  $$\rightarrow \text{Con}\left(\text{ZFC} + \begin{array}{l}\text{There are infinitely}\\ \text{many Woodin cardinals}\end{array}\right)$$

3. It seems that every natural extension of ZFC is equiconsistent with an extension axiomatized by something like large cardinal axioms. These natural consistency strengths seem to be wellordered! This wellorder corresponds to the inclusion order on the set of Π^0_1 (or in fact, arithmetical, or even Σ^1_2) consequences of the theories in question.

   At the level of Σ^1_2 sentences, we know of only one road upward, and the large cardinal hypotheses are its central markers.

4. For T a theory and Γ a set of sentences, let

$$(\Gamma)_T = \{\varphi \in \Gamma \mid T \vdash \varphi\}.$$

   For T and U natural theories of consistency strength at least that of "There are n Woodin cardinals with a measurable above", we have

$$(\Pi^0_1)^T \subseteq (\Pi^0_1)^U \leftrightarrow (\Sigma^1_n)^T \subseteq (\Sigma^1_n)^U.$$

5. Any natural theory of consistency strength at least that of PD actually implies PD. For example, the Proper Forcing Axiom implies PD. So does the existence of a homogeneous saturated ideal on ω_1.

   At the level of Σ^1_n sentences, we know of only one road upward, and large cardinals are its central markers; moreover this road goes through PD.

6. Large cardinal axioms have proved very fruitful in deciding the questions about projective sets of classical descriptive set theory. The theory of projective sets of gets from large cardinal axioms extends in a natural way the theory of low-level projective sets one gets from ZFC alone. This theory is axiomatized by projective determinacy (PD):

   Theorem 1 (Martin, Woodin, S.) *The following are equivalent:*

   (a) PD ,

   (b) For all $n < \omega$, every Σ^1_n consequence of ZFC + "there are n Woodin cardinals" is true.

   Thus PD is the "instrumentalist's trace" of Woodin cardinals in the language of second order arithmetic.

   The theory of projective sets one gets from large cardinals is much more natural, and potentially useful, than the theory one gets from $V = L$.

7. **Large cardinal axioms seem to decide all natural questions in the language of second order arithmetic.** There is metamathematical evidence of this completeness in the fact that no sentence in the language of second order arithmetic can be shown independent of existence of arbitrarily large Woodin cardinals by forcing:

   Theorem 2 (Woodin) *Suppose that over every set belongs to an iterable inner model satisfying "there are ω Woodin cardinals"; then if M and N are set-generic extensions of V, we have*

   $$L(\mathbb{R})^M \equiv L(\mathbb{R})^N.$$

8. There is only one theory with this kind of "generic completeness":

   Theorem 3 (Woodin, S.) *Suppose that whenever M and N are set generic extensions of V, we have $L(\mathbb{R})^M \equiv L(\mathbb{R})^N$; then every set belongs to an iterable inner model satisfying "there are ω Woodin cardinals".*

Generic Absoluteness and CH

A Σ^2_n sentence is one of the form $(V_{\omega+2}, \in) \models \varphi$, where φ is Σ_n in the Levy hierarchy.

CH is (equivalent to) a Σ^2_1 sentence. There are no generic absoluteness theorems at the Σ^2_1 level:

Theorem 4 (Levy, Solovay) *The current large cardinal axioms are* preserved by small forcing.

Corollary 5 *Let A be one of the current large cardinal axioms, and suppose $V \models A$; then there are set generic extensions M and N of V which satisfy A, such that $M \models CH$ and $N \models \neg CH$.*

Question Are all large cardinal axioms preserved by small forcing?

Perhaps there is some family natural extrapolations from our large cardinal hypotheses which mark still higher consistency strengths and which decide CH.

This would be a solution closest to the sort Godel envisaged.

For a possible parallel, imagine that A has only been able to conceive of the weaker large cardinal hypotheses, and so takes it to be a general feature of large cardinal hypotheses that they relativise to L, as well as are preserved under small forcing. On this basis he concludes that large cardinal hypotheses will never decide the Lebesgue measurability of projective sets.

In this case, the fact that A's concept of large cardinals was too restrictive would show up in the many consistency questions he could not answer; for example, Con(PD).

Question: Is it consistent that CH holds, and every set of reals definable from real parameters over $(V_{\omega+2}, \in)$ is determined, and admits a scale which is so definable.

Conditional generic absoluteness

We look now at another alternative, which keeps more features of the current situation: all "natural" consistent statements can be forced under some large cardinal statements, and large cardinals are preserved by small forcing.

One can still hope to found a "complete" theory of $(V_{\omega+2}, \in)$ on principles which, under some large cardinal hypothesis, are true in a set-generic extension of V. In effect, one must say which sort of generic extension one wants to take as one's "reference point" V. In the most appealing scenario, it will be possible to say "everything" about one's reference point, and the metamathematical evidence of that will be a *conditional* generic absoluteness theorem. The prototype at the Σ^2_1 level is:

Theorem 6 (Woodin) *Suppose $V \models$ "There are arbitrarily large measurable Woodin cardinals". Let M and N be set-generic extensions of V satisfying CH; then M and N are Σ^2_1-equivalent.*

Thus CH is *generically complete* (or Ω^*-*complete*) at the Σ^2_1-level, in the presence of large cardinals. It is also generically consistent (Ω^* *consistent*).

One needs more than CH to go on to Σ^2_2:

Theorem 7 (Abraham, Shelah) *For any of the current large cardinal axioms A, if $V \models A$, then there are set generic extensions M and N of V which satisfy CH such that M satisfies "there is a Σ^2_2 wellorder of the reals", while N satisfies "every HOD$(\mathbb{R})$ set of reals is Lebesgue measurable".*

Question: Assume there are arbitrarily large supercompact cardinals, and let M and N be set-generic extensions of V satisfying $\diamondsuit$; must M and N be Σ^2_2-equivalent?

Question: Does some large cardinal hypothesis imply that the theory of $(V_{\omega+2}, \in)$ in $V^{\mathrm{col}(\omega_1, <\kappa)}$, for say κ the first inaccessible, is not changed by set forcing?

More generally, can one find recursively axiomatizable theories T_n such that (under some large cardinal assumption)

- T_n is true in some set-generic extension of V, and
- any two set-generic extensions of V satisfying T_n are Σ^2_n-equivalent, and
- $T_n \subseteq T_{n+1}$, for all n.

(The formulation suggests that the axioms of T_n might be approximately Σ^2_n themselves.)

Under our present scenario (i.e., all large cardinal hypotheses are preserved by small forcing, and every interesting (or even just true) theory can be set-forced in the presence of some sufficiently strong large cardinal hypotheses):

finding some such T_n extendible to theories T_α of arbitrarily high V_α is all there is to deciding the theory of $(V_{\omega+2}, \in)$.

For if S is any theory of $(V_{\omega+2}, \in)$ which can be forced over universes satisfying large cardinal hypothesis A via a poset $P \in V_\alpha$, then S can be interpreted by the believer in T_α (plus large cardinals) as the theory of $(V_{\omega+2}, \in)$ after forcing with P. That is, large cardinal plus $\bigcup_\alpha T_\alpha$ *interprets all generically consistent alternatives*.

Note that in this scenario, there may well be more than one such sequence of theories T_α. Any two such sequences will be “generically bi-interpretable”. Developing one would be the same activity as developing the other, so that there would be no significant behavioral difference between adopting one and adopting the other. The situation would be like V=L versus V=L[G].

Finding any such sequence of theories would solve the Continuum Problem. The existence of sequences disagreeing on CH would be no more an indication of incompleteness that the existence of such sequences extending AFA on the one hand, and ZFC on the other.

The Ω-conjecture

In the arguments we have now, generic absoluteness is always enforced by Hom_∞ sets. The Ω-conjecture states that this is a general feature of all generic absoluteness. That, in turn, implies that there can be no generic absoluteness theorems of the sort we have suggested looking for.

Psuedo-definitions:

(a) $T \overset{\Omega^*}{\vdash} \varphi$ iff for any G set-generic over V and any α, $(V_\alpha)^{V[G]} \models T \Rightarrow (V_\alpha)^{V[G]} \models \varphi$.

(b) $T \overset{\Omega}{\vdash} \varphi$ iff there is a Hom_∞ set which guarantees that $T \overset{\Omega^*}{\vdash} \varphi$.

In this language, we have been asking for a recursively axiomatized, Ω^*-complete, Ω^*-consistent theory of $(V_{\omega+2}, \in)$.

Woodin has shown that if Ω^*-consequence is strengthened to Ω-consequence, then these demands become too strong.

Theorem 8 (Woodin) *(a) If T is recursively axiomatizable, then*

$$\{\varphi \mid T \overset{\Omega}{\vdash} (V_{\omega+2}, \in) \models \varphi\} \textit{ is } \Sigma^2_3.$$

(b) If T is recursively axiomatizable and Ω-consistent, then for some φ

$$(V_{\omega+2}, \in) \models \varphi$$

but

$$\neg(T \overset{\Omega}{\vdash} (V_{\omega+2}, \in) \models \varphi.$$

Thus Ω-completeness is too much to ask for in our theory of $(V_{\omega+2}, \in)$.

Ω-conjecture: For any T and φ,

$$T \overset{\Omega^*}{\vdash} \varphi \Rightarrow T \overset{\Omega}{\vdash} \varphi.$$

There is evidence for the Ω-conjecture, some of it in the form "if there are inner models for cardinal past supercompact which are canonical in the same way the inner models we know are canonical, then the Ω-conjecture holds."

If the Ω conjecture is true, then Ω^*-completeness is too much to ask for in our theory of $(V_{\omega+2}, \in)$. The metamathematical indicator for the sort of "practical completeness" we seek which worked for the theory of $(V_{\omega+1}, \in)$ cannot work for $(V_{\omega+2}, \in)$.

In this case, it is not clear what weaker demand we might put on the theory of our "reference point", or whether we should be seeking one at all. It might be appropriate consider set theory as the study of a "Kripke model" of universes, with accessibility being given by generic extension, in which no particular reference universe stands out as a root.