Is there really any evidence that the Continuum Hypothesis has no answer[1]?

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[1] annotated

Axioms of Set Theory

The 20th century choice:

- The Zermelo-Frankel axioms together with the Axiom of Choice.
  - These are the ZFC axioms.

One of the most well known problems of Set Theory (and certainly the oldest):

Cantor's Continuum Hypothesis: Suppose that

$$X \subset \mathbb{R}$$

is an infinite set. Then either

$$\mathrm{cardinality}(X) = \mathrm{cardinality}(\mathbb{N})$$

or

$$\mathrm{cardinality}(X) = \mathrm{cardinality}(\mathbb{R}).$$

The first result concerning CH was obtained by Gödel.

Theorem 1 (Gödel, 1938) *Assume* ZFC *is consistent. Then so is* $\mathrm{ZFC} + \mathrm{CH}$. □

The modern era of set theory began with Cohen's discovery of the method of *forcing* and his application of this new method to show:

Theorem 2 (Cohen, 1963) *Assume* ZFC *is consistent. Then so is*

$$\mathrm{ZFC} + \text{“CH is false”}. \qquad \square$$

Thus what is perhaps the most fundamental question one can naturally ask about infinite sets is not (formally) solvable from the axioms.

There are several interpretations of this that have been proposed.

1) There are new axioms to be discovered and validated which settle CH.

2) There are *no* such new axioms to be discovered and validated because the conception of general sets is *inherently vague*.

How does forcing work?

Given a universe of sets V there correspond to the (non-atomic) Boolean algebras, $\mathbb{B}$, of V new universes of Set Theory, denoted $V^{\mathbb{B}}$. These are “virtual” universes which contain V.

For a given formal sentence ϕ, whether or not ϕ is true in $V^{\mathbb{B}}$ is a property of the Boolean algebra, $\mathbb{B}$, in the initial universe V. It is this fact that is the key feature of forcing.

The fundamental difficulty is that for many questions, such as CH, there always exists a Boolean algebra $\mathbb{B}$ such that the question is true in $V^{\mathbb{B}}$ and there always exists a Boolean algebra $\mathbb{B}$ such that the question is false in $V^{\mathbb{B}}$.

This is a PROBLEM.

Perhaps the foundational difficulties created by forcing can be mitigated by simply adopting a "many worlds" view[1]. For example given a sentence, ϕ, one simply asks if

$$\langle \mathcal{P}(\mathbb{R}), \mathbb{R}, \cdot, +, \in \rangle \models \phi,$$

holds in $V^{\mathbb{B}}$ for some Boolean algebra, $\mathbb{B}$.

In this particular view the analysis of CH is finished: there are worlds in which it holds and there are worlds in which it fails.

But is this really a viable view? Or is it formalism in disguise?

[1] I really only intend to investigate the potential foundational view:

Given a sentence, ϕ, one simply asks if

$$\langle \mathcal{P}(\mathbb{R}), \mathbb{R}, \cdot, +, \in \rangle \models \phi,$$

holds in $V^{\mathbb{B}}$ for some Boolean algebra, $\mathbb{B}$.

This looks like some sort of many-worlds view to me, but exactly how this view is incorporated into a coherent foundational view is not really clear to me.

The cumulative hierarchy of sets:

1. $V_0 = \emptyset$.
2. $V_{\alpha+1} = \mathcal{P}(V_\alpha)$.
3. If β is a limit ordinal then

$$V_\beta = \cup \left\{ V_\alpha \,\middle|\, \alpha < \beta \right\}.$$

It is a consequence of the axioms, ZFC, that every set belongs to V_α for some ordinal α.

Given a Boolean algebra $\mathbb{B}$, $V_\alpha^{\mathbb{B}}$ denotes V_α as *defined* in $V^{\mathbb{B}}$.

(Since V is contained in $V^{\mathbb{B}}$, α is an ordinal in the sense of $V^{\mathbb{B}}$).

Suppose T is a theory in the language of Set Theory and ϕ is a sentence. Define

$$T \models_\Omega \phi$$

if for all Boolean algebras, $\mathbb{B}$, and for all ordinals α, if

$$V_\alpha^{\mathbb{B}} \models T$$

then $V_\alpha^{\mathbb{B}} \models \phi$.

This defines[1] Ω-logic which of course is simply the logic given by the many-worlds view–except we have enlarged our family of worlds to include the rank initial segments.

[1] This is Ω^*-logic in some previous accounts

There is a remarkable fact:

Theorem 3 *Suppose that there is a proper class of Woodin cardinals, T is a set of sentences and that ϕ is a sentence. Then for each complete Boolean algebra $\mathbb{B}$,*

$$V \models \text{“}T \models_\Omega \phi\text{”}$$

if and only if

$$V^{\mathbb{B}} \models \text{“}T \models_\Omega \phi\text{”}. \qquad \square$$

In short:

> **Ω-logic *cannot* be affected by passing from V to $V^{\mathbb{B}}$ for any choice of $\mathbb{B}$.**

This seems to offer evidence in favor of the many-worlds view. Or does it?

To better understand Ω-logic we need to find a corresponding notion of proof[1].

In fact there is a natural candidate for the definition of an "Ω-proof".

The definition requires a generalization of the Borel sets.

$L(\mathbb{R})$ denotes the smallest inner model of ZF which contains the reals and the ordinals.

The sets in

$$\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$$

are a natural generalization of the Borel sets–but we will need to generalize still further and define a *much larger* class of sets of reals.

[1] if there is one

A set of reals, $A \subseteq \mathbb{R}$, is *universally Baire* if for every compact Hausdorff space, Ω, and for every continuous function

$$F : \Omega \to \mathbb{R},$$

the set $\left\{x \in \Omega \,\middle|\, F(x) \in A\right\}$ has the property of Baire: i.e. there exists an open set $O \subseteq \Omega$ such that the symmetric difference,

$$\left\{x \in \Omega \,\middle|\, F(x) \in A\right\} \triangle O,$$

is meager.

If $A \subseteq \mathbb{R}$ then $L(A, \mathbb{R})$ denotes the smallest inner model of ZF which contains the reals and the ordinals and which contains the set A.

The class of sets of reals that we need in order to define the proof relation,

$$T \vdash_\Omega \phi$$

for Ω-logic is the collection of all universally Baire sets $A \subseteq \mathbb{R}$ such that:

1. $L(A, \mathbb{R}) \models \mathrm{AD}^+$;
2. Every set in $\mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R})$ is universally Baire.

In the presence of a proper class of Woodin cardinals these are just the universally Baire sets of reals.

These universally Baire sets provide a natural notion of an Ω-proof.

The definition requires the following technical notion:

> *Suppose that $A \subseteq \mathbb{R}$ is universally Baire and that M is a countable transitive model of* ZFC.
>
> *The set M is A-**closed** if for all countable transitive models, N, if N is a set generic extension of M then*
>
> $$A \cap N \in N.$$

Suppose that T is a theory and that ϕ is a sentence. Then[1]

$$T \vdash_\Omega \phi$$

if there exists a set $A \subseteq \mathbb{R}$ such that

1. $L(A, \mathbb{R}) \models \mathrm{AD}^+$,
2. every set in $\mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R})$ is universally Baire,
3. for all countable transitive A-closed sets M,

$$M \models \text{“}T \models_\Omega \phi\text{”}.$$

[1] the definition here is slightly changed from that in some previous accounts. The changes result from having defined $\models_\Omega$ instead of introducing Ω^*-logic, and from giving the definition without assuming there is a proper class of Woodin cardinals

Thus if there exists a proper class of Woodin cardinals, then[1]

$$T \vdash_\Omega \phi$$

if there exists a universally Baire set $A \subseteq \mathbb{R}$ such that for all countable transitive A-closed sets M,

$$M \models \text{“}T \models_\Omega \phi\text{”}.$$

[1]this is the definition of $T \vdash_\Omega \phi$ assuming there is a proper class of Woodin cardinals.

The universally Baire set, A, is in essence, the "Ω-proof" and there is natural notion of the length of this proof which is given by the ordinal rank of A in a transfinite hierarchy of complexity (the Wadge hierarchy).

With this definition of length, one can define the usual sorts of Gödel sentences etc.

Thus in many respects this notion of proof for Ω-logic is a natural transfinite generalization of the classical notion of proof for first order logic.[1]

[1]One might take issue with this being a true notion of proof. It is certainly not a finitary notion. Here is a motivation for the definition (taken from; *"Set Theory after Russell"*, to appear)

First for each $a \in \mathbb{R}$, for each set of sentences T, and for each formula $\phi(x_0)$ one can generalize the definition $\models_\Omega$ in a natural fashion and define the relation $T \models_\Omega \phi[a]$. More precisely $T \models_\Omega \phi[a]$ if for all complete Boolean algebras, $\mathbb{B}$, for all ordinals α, if

$$V_\alpha^{\mathbb{B}} \models T$$

then $V_\alpha^{\mathbb{B}} \models \phi[a]$.

Define a set $A \subset \mathbb{R}$ to be Ω-*finite* if there exists a formula $\phi(x_0)$ such that

$$A = \left\{a \in \mathbb{R} \,\middle|\, \emptyset \models_\Omega \phi[a]\right\}$$

and such that for all complete Boolean algebras, $\mathbb{B}$, the following holds in $V^{\mathbb{B}}$:

For all $a \in \mathbb{R}$ *either* $\emptyset \models_\Omega \phi[a]$ *or* $\emptyset \models_\Omega (\neg\phi)[a]$.

Suppose that $\emptyset \models_\Omega \phi$. Then by analogy with first order logic, there should exist a set $A \subseteq \mathbb{R}$ such that A is Ω-finite and such that for all countable transitive models, M, of ZFC, if M is suitably closed under A then

$$M \models \text{“}\emptyset \models_\Omega \phi\text{”}.$$

Given the definition of the relation, $\emptyset \models_\Omega \phi$, the requirement that M be suitably closed under A should be:

For all Boolean algebras, $\mathbb{B}_M \in M$, if $\mathbb{B}$ is the completion of $\mathbb{B}_M$ and if $G \subseteq \mathbb{B}$ is V-generic then

$$A_G \cap M[G \cap \mathbb{B}_M] \in M[G \cap \mathbb{B}_M]$$

where A_G is the set of $a \in \mathbb{R}^{V[G]}$ such that

$$V[G] \models \text{“}\emptyset \models_\Omega \phi_A[a]\text{”},$$

and where $\phi_A(x_0)$ is a formula witnessing that A is Ω-finite.

In short, the Ω-finite sets should suffice to “witness” the relation $\emptyset \models_\Omega \phi$, at least for countable transitive models.

Theorem (ZFC) *Suppose that there is a proper class of Woodin cardinals and that $A \subseteq \mathbb{R}$ is Ω-finite.*

Then every set in $\mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R})$ is universally Baire and $L(A, \mathbb{R}) \models \mathrm{AD}^+$. □

Now suppose that there is a proper class of Woodin cardinals. It follows that for each Ω-finite set A, if M is a countable transitive set such that

$$M \models \mathrm{ZFC}$$

and such that M is A-closed then M is suitably closed under A in the sense defined above.

This is the motivation for the definition of $\vdash_\Omega$.

Theorem 4 (Ω-soundness) (ZFC) *Suppose that T is a set of sentences, that ϕ is a sentence, and that $T \vdash_\Omega \phi$. Then $T \models_\Omega \phi$.* $\square$

In fact there is an equivalent definition of $\vdash_\Omega$ which does not refer to the universally Baire sets at all. In this approach, which is much more complicated to define, the proofs are "fine-structural" objects in a hierarchy which generalizes that of the L_α. For infinitary language, $\mathcal{L}_{\omega_1,\omega}$, if ϕ is valid then a proof of ϕ must occur in L_α for some recursive ordinal, α. One could say that if ϕ is valid then this is "certified" by L_α for some recursive ordinal, α. Similarly if

$$\emptyset \models_{\beta\text{-logic}} \phi$$

then this is certified by L_α for some countable ordinal α.

Now if $\emptyset \vdash_\Omega \phi$, so by soundness $\emptyset \models_\Omega \phi$, then there is a structure M in a hierarchy which generalizes the heirarchy of the L_α which certifies that $\emptyset \models_\Omega \phi$. So an alternate definition of $\vdash_\Omega$ could be given by defining this hierarchy.

We now come the key conjecture, this is the Ω Conjecture:

> *Suppose that there exists a proper class of Woodin cardinals. Then for each sentence ϕ,*
>
> $$\emptyset \models_{\Omega} \phi$$
>
> *if and only if*
>
> $$\emptyset \vdash_{\Omega} \phi.$$

The theorem that this conjecture is true would simply be the *Completeness Theorem* for Ω-logic.

It is important to note that if there exists a proper class of Woodin cardinals then for all complete Boolean algebras, $\mathbb{B}$,

$$V^{\mathbb{B}} \models \Omega \text{ Conjecture}$$

if and only if the Ω Conjecture holds in V.

Therefore it is very unlikely that the problem of the Ω Conjecture is unsolvable in the same fashion that CH is unsolvable.

Suppose that the Ω Conjecture holds in V (and there exists a proper class of Woodin cardinals). Then:

- One can precisely define the large cardinal hierarchy and quantify its influence;

Further:

- The theory of $\mathcal{P}(\omega_1)$ can be completely unambiguous – in the strong sense that for some sentence Ψ_0 and for all sentences ϕ,

  $$\langle \mathcal{P}(\omega_1), \omega_1, +, \cdot, \in \rangle \models \phi$$

  if and only if

  $$\{\Psi_0\} \models_\Omega \text{“}\langle \mathcal{P}(\omega_1), \omega_1, +, \cdot, \in \rangle \models \phi\text{”}$$

  But if this happens then CH *is false*;

because:

- The theory of $\mathcal{P}(\mathbb{R})$ *cannot* be unambiguous in this strong sense; i.e there *cannot* exist a sentence Ψ_0 such that for all sentences ϕ,

$$\langle \mathcal{P}(\mathbb{R}), \mathbb{R}, +, \cdot, \in \rangle \models \phi$$

  if and only if

$$\{\Psi_0\} \models_\Omega \text{“}\langle \mathcal{P}(\mathbb{R}), \mathbb{R}, +, \cdot, \in \rangle \models \phi\text{”}$$

Two possible futures

Future Possibility I

The Ω *Conjecture is false.*

There are a variety of ways this might happen.

In one extreme, the set

$$\left\{\phi \,\middle|\, \emptyset \models_{\Omega} \phi\right\},$$

might be recursively equivalent to the complete Π_2 set of integers and so be as *complicated* as possible.

In this case one might reasonably take the view that at least for questions about $\mathcal{P}(\mathbb{R})$:

> *The only questions which are meaningful are those which are* $\models_{\Omega}$ *valid or those for which the negation is* $\models_{\Omega}$ *valid.*

This of course is the many-worlds view[1].

There are no current examples of statements whose truth is known not to be resolvable on the basis of this view – allowing, of course, that the resolution may be that the question has no answer as would be the case for CH, and allowing for a resolution based on large cardinal axioms.

[1]there is the problem of how to incorporate this into a coherent foundational view, but that is a separate problem

But, it might *also* be the case that for some sentence, Ψ_0, the sentence Ψ_0 holds in V and for all Π_2 sentences, ϕ,

$$V \models \phi$$

if and only if

$$\{\Psi_0\} \models_\Omega \phi.$$

This is the most natural way that the set

$$\left\{\phi \,\middle|\, \emptyset \models_\Omega \phi\right\},$$

be recursively equivalent to the complete Π_2 set of integers.

A technical digression.

Notice that the sentence Ψ_0 has the property that for all Σ_2 sentences, ϕ_1 and ϕ_2, if

$$\{\Psi_0, \phi_1\}$$

is $\models_\Omega$-*satisfiable*, (i.e. if $\{\Psi_0\} \not\models_\Omega \neg\phi_1$), and if

$$\{\Psi_0, \phi_2\}$$

is $\models_\Omega$-satisfiable then

$$\{\Psi_0, \phi_1, \phi_2\}$$

is $\models_\Omega$-satisfiable.

If this holds with real parameters in V and in $V^{\mathbb{B}}$ for all $\mathbb{B}$, then every OD set of reals is universally Baire.

Theorem 5 *Suppose that there is a proper class of Woodin cardinals and that every* OD *set,* $A \subset \mathbb{R}$*, is universally Baire.*

Then the Ω *Conjecture holds in* HOD. $\square$

In the other extreme, the set

$$\left\{\phi \middle| \emptyset \models_{\Omega} \phi\right\},$$

might always be *simple* to define, more precisely that for all complete Boolean algebras, $\mathbb{B}$, this set of integers is definable, without parameters, in the structure,

$$\left(\langle \mathcal{P}(\omega_1), \omega_1, \cdot, +, \in \rangle\right)^{V^{\mathbb{B}}}.$$

While this also implies that the Ω Conjecture is false, the meta-mathematical issues in this case are similar to those that arise if the Ω Conjecture is true. This of course is the second possible future.

Regarding **Future Possibility I** there is an important point:

> *It is known that the Ω Conjecture is relatively consistent with a proper class of Woodin cardinals. However it is* not *known if the Ω Conjecture can consistently fail.*

Future Possibility II

The Ω *Conjecture is true*
(*and there exists a proper class of Woodin cardinals*).

Then the set

$$\left\{\phi \,\middle|\, \emptyset \models_\Omega \phi\right\}$$

cannot be recursively equivalent to the complete Π_2 set of integers. In fact this set of sentences (integers) is definable in the structure,

$$\langle \mathcal{P}(\mathbb{R}), \mathbb{R}, +, \cdot, \in \rangle,$$

without parameters[1].

In this case the many-worlds view is no more viable than formalism.

[1]But not uniformly! The actual definition depends on the specific nature of V

The challenge then is this:

Exhibit a sentence ϕ such that assertion:

$$\langle \mathcal{P}(\mathbb{R}), \mathbb{R}, +, \cdot, \in \rangle \models \phi$$

is *true* but not $\models_\Omega$ valid.

If there is no such sentence then the only "truths" for the standard structure for *Second Order Analysis*,

$$\langle \mathcal{P}(\mathbb{R}), \mathbb{R}, +, \cdot, \in \rangle,$$

are those which are $\models_\Omega$ valid *and this set is too simple*.

So there must be such a sentence!

But any such sentence is as unsolvable as CH;

- it can always be forced to hold;
- it can always be forced to fail.

Validating such a truth would be an instance of a new phenomenon. If in addition, the assertion

$$\text{“}\langle \mathcal{P}(\mathbb{R}), \mathbb{R}, +, \cdot, \in \rangle \models \phi\text{” is } \textit{not} \models_\Omega \text{ valid}$$

is *provable* from ZFC then the validation would be an instance of something genuinely new–fundamentally different from the discovery and validation of the axiom of *Projective Determinacy*.

This new phenomenon could well lead to a resolution of CH as well.

There are many natural candidates for ϕ other than CH and its negation, for example:

There is a Σ^2_1*-definable wellordering of* $\mathbb{R}$;

and

Every Σ^2_1*-definable set of reals is determined*;

(and their negations) are each candidates (and less controversial than CH).

In summary:

> **The Ω Conjecture will have an answer. Further the answer will have profound consequences for the foundations of Set Theory.**

Moreover:

> **If the Ω Conjecture is true then the only evidence we currently have that** CH **is ambiguous (i.e. forcing), is not evidence for the ambiguity of** CH **at all!**