Ordinal Analyses and Large Cardinals

Abstract

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One of the major topics of proof theoretical research are ordinal analyzes of axiom system. Ordinal analyzes go back to Gentzen's 1943 paper [1] in which he proved that the order–type of any elementarily definable ordering whose well–foundedness is provable from the axioms of Peano arithmetic has an order type less than $\varepsilon_0 := \min \{\alpha | \ \omega^\alpha = \alpha\}$ while conversely every ordinal less than ε_0 can represented by an elementarily definable ordering whose well–foundedness is provable from the Peano axioms. Since then we define the proof theoretical ordinal of a theory T as the supremum of the order–types of all elementarily definable orderings whose well–foundedness is provable from T. The computation of the proof theoretical ordinal of a theory T is called an *ordinal analysis* for T.
There is a variety of alternative definitions of the proof theoretical ordinal of a theory. For theories in the language of set theory the proof theoretical ordinal of a theory T can be defined as the least ordinal α such that the αth level of the constructible hierarchy is closed under the ω_1^{CK}–recursive functions whose totality is provable in T.
The proof theoretical ordinal of a (reasonable) theory is always a constructive ordinal and is therefore effectively describable. Finding the effective description for the proof theoretical ordinals of strong theories (strong in the proof theoretical sense) is one of the big challenges. It has turned out that large cardinals are helpful in finding these descriptions. This can be put so far, that we have corresponding proof theoretic ordinals for a few (mildly) large cardinals. It is an interesting feature that many of the closure properties of these cardinals are reflected by their corresponding proof theoretical ordinals.

References

[1] G. GENTZEN, *Beweisbarkeit und Unbeweisbarkeit von Anfangsfällen der transfiniten Induktion in der reinen Zahlentheorie*, **Mathematische Annalen**, vol. 119 (1943), pp. 140–161.

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