Dedekind Completeness & Denumerability

Let $\mathcal{X} = \langle X \preceq \rangle$ be a totally ordered set. Then $\mathcal{X}$ is said to be **Dedekind complete** if and only if for each nonempty subset $Z$ of $X$, there exists an element $a$ in $X$, called the **supremum of $Z$ (in $X$)**, or sup $Z$ for short, such that

(i) $z \preceq a$ for all $z$ in $Z$, and

(ii) if $b$ in $X$ is such that $z \preceq b$ for all $z$ in $Z$, then $b = a$.

A set $Y$ is said to be **denumerable** if and only if there exists a one-to-one function $f$ from $\mathbb{I}^+$ onto $Y$.

Let $\langle X, \preceq \rangle$ be a totally ordered set and $f$ a function from a subset $Y$ of $X$ into $X$. Then $f$ is said to be **$\preceq$-strictly increasing** if and only if for all $x$ and $y$ in $X$, if $x \prec y$ then $f(x) \prec f(y)$. 
Continuum

\( \langle X, \preceq \rangle \) is said to be a **continuum** if and only if the following four statements are true:

1. **Total ordering**: \( \preceq \) is a total ordering on \( X \).
2. **Unboundedness**: \( \langle X, \preceq \rangle \) has no \( \preceq \)-greatest or \( \preceq \)-least element.
3. **Denumerable density**: There exists a denumerable subset \( Y \) of \( X \) such that for each \( x \) and \( z \) in \( X \), if \( x \prec z \) then there exists \( y \) in \( Y \) such that \( x \prec y \) and \( y \prec z \).
4. **Dedekind completeness**: \( \langle X, \preceq \rangle \) Dedekind complete.
**Theorem (Cantor 1895):** *(existence)* $\mathcal{X} = \langle X, \leq \rangle$ is a continuum if and only if $\mathcal{X}$ isomorphic to $\langle \mathbb{R}^+, \leq \rangle$.

**Theorem:** *(uniqueness)* Let $\mathcal{X} = \langle X, \leq \rangle$ be a continuum and $\mathcal{F}$ be the set of isomorphisms of $\mathcal{X}$ onto $\langle \mathbb{R}^+, \leq \rangle$. Then for each $f$ in $\mathcal{F}$ $\mathcal{F} = \{ g \ast f \mid g \text{ is a strictly increasing function from } \mathbb{R}^+ \text{ onto } \mathbb{R}^+ \}$. 
Scale Families

Let $Y$ be a nonempty set. A **scale family** (or **scale**) on $Y$ is a nonempty set of functions from $Y$ onto a subset of $\mathbb{R}$.

Let $\mathcal{F}$ be a scale family on $Y$. Then the elements of $\mathcal{F}$ are called **measuring functions**.
Some Scale Types

\( \mathcal{F} \) is said to be a **ratio scale** if and only if

1. \( \mathcal{F} \) is a scale family, and
2. for any \( f \) in \( \mathcal{F} \),
   \[
   \mathcal{F} = \{ rf \mid r \in \mathbb{R}^+ \}.
   \]

\( \mathcal{F} \) is said to be an **interval scale** if and only if

1. \( \mathcal{F} \) is a scale family, and
2. for any \( f \) in \( \mathcal{F} \),
   \[
   \mathcal{F} = \{ rf + s \mid r \in \mathbb{R}^+ \& s \in \mathbb{R} \}.
   \]

\( \mathcal{F} \) is said to be an **ordinal scale** if and only if

1. \( \mathcal{F} \) is a scale family, and
2. for any \( f \) in \( \mathcal{F} \), \( \mathcal{F} = \{ g \ast f \mid g \) is a strictly increasing function from \( \mathbb{R}^+ \) onto \( \mathbb{R}^+ \} \).
Extend the modern representational approach by replacing extensive structures with structures of the form $\langle X, \preceq, \oplus \rangle$ satisfying the same axioms as extensive structures, except possibly associativity. Such structures are called PCSs (with a partial operation), or just PCSs when the operation is closed.

In particular, Narens & Luce showed that the major results of *Foundations of Measurement, Vol. 1* could be generalized using PCSs with partial or closed operations in place of extensive structures.
Existence & Uniqueness of Representations

\[ \mathcal{X} = \langle X, \preceq, \oplus \rangle \] is a PCS.

Narens & Luce’s formulation of PCSs with partial or closed operations used the following axiom: half elements: for each \( x \) there exists \( y \) such that \( y \oplus y = x \).

With half-elements they showed that there exist a structure \( \mathcal{N} = \langle R, \leq, \oplus \rangle, R \subseteq \mathbb{R}^+ \), such that

- (existence) there exists an isomorphism from \( \mathcal{X} \) onto \( \mathcal{N} \), and
- (uniqueness) for all isomorphisms \( \varphi \) and \( \psi \) from \( \mathcal{X} \) onto \( \mathcal{N} \), if for some \( a \) in \( X \), \( \varphi(a) = \psi(a) \), then \( \varphi = \psi \).
\( \mathcal{X} = \langle X, \preceq, \oplus \rangle \) is a PCS with a partial or closed operation. Then \( \alpha \) is said to be a \textbf{symmetry} of \( \mathcal{X} \) if and only if \( \alpha \) is a function from \( X \) onto itself such that for all \( x \) and \( y \) in \( X \),

\[
x \preceq y \iff \alpha(x) \preceq \alpha(y)
\]

and

\[
\alpha(x \oplus y) = \alpha(x) \oplus \alpha(y).
\]

\( \mathcal{X} \) is said to be \textbf{homogeneous} if and only if for all \( a \) and \( b \) in \( X \), there exists a symmetry \( \alpha \) of \( \mathcal{X} \) such that \( \alpha(a) = b \).
Cohen & Narens (1979)

Cohen: Half-elements not needed. His method of proof used symmetries.

Narens: Homogeneous PCSs are ratio scalable.

A PCS $\mathcal{X} = \langle X, \preceq, \oplus \rangle$ is said to be **homogeneous** if and only if for each $x$ and $y$ in $X$ there exists a symmetry $\alpha$ of $\mathcal{X}$ such that

$$\alpha(x) = y.$$
n-copy operator

\( \mathcal{X} = \langle X, \preceq, \oplus \rangle \) is a PCS.

Let \( 1x = x \) and for each positive integer \( n \), let
\( (n + 1)x = (nx) \oplus x \). \( nx \) is called the \textit{n-copy operator} of \( \mathcal{X} \).

**Theorem** The following two statements are equivalent:

1. \( \mathcal{X} \) is homogeneous.

2. For each positive integer \( n \), the \textit{n-copy operator} is a symmetry of \( \mathcal{X} \).
Examples of PCSs

(1) $\langle \mathbb{R}^+, \leq, + \rangle$.

(2) $\langle \mathbb{R}^+, \leq, \oplus_1 \rangle$, where $r \oplus_1 s = r + s + r^{\frac{1}{3}} s^{\frac{2}{3}}$.

(3) $\langle \mathbb{R}^+, \leq, \oplus_2 \rangle$, where $r \oplus_1 s = r + s + r^2 s^2$.

(1) and (2) are homogeneous with multiplications by positives reals as their symmetries.

(3) has the identity as its only symmetry.
Unit Representations

**Theorem** Suppose $\mathcal{X} = \langle X, \preceq, \oplus \rangle$ a homogeneous PCS, $R \subseteq \mathbb{R}^+$, and $\varphi$ is an isomorphism of $\langle X, \preceq, \oplus \rangle$ onto $\langle R, \leq, \odot \rangle$. Then the following two statements hold:

1. **Unit Representation**: There exists a function on $R$ such that for all $r$ and $s$ in $R$,

   $$ r \odot s = s \cdot f \left( \frac{r}{s} \right). $$

2. $\langle X, \preceq, \oplus \rangle$ is an extensive structure if and only if $f(x) = 1 + x$.

PCSs with unit representations produce ratio scale representations and they can be used to generalize extensive measurement and its uses.
(1) Generalized the Cohen & Narens to arbitrary qualitative structures
\[ \mathcal{X} = \langle X, R_1, \ldots, R_i \rangle \]
to produce an even more general theory of ratio scale measurement.

(2) Developed a general theory for derived measurement scales for functions of several ratio scalable variables.
Luce & Cohen (1983)

Extended Narens (1981a) to situations involving interval scalable structures, e.g., structures isomorphic to

\[ \langle \mathbb{R}, \leq, \oplus \rangle, \text{ where } r \oplus s = \frac{r + s}{2}. \]

(If an interval scalable structure is properly measurable by \( \varphi \), then all other proper measures are of the form \( r\varphi + s \), where \( r > 0 \) and \( s \) is real.)
Luce & Narens (1985)

Synthesized, reformulated, and extended the results of the previous mentioned articles concerning PCSs and its generalizations.

Applied the theory to utility theory, and provided a measurement-theoretic formulation of rank dependent utility theory for two outcome gambles.

Luce (2000) **Utility of Gains and Losses**