Chapter 5: Probability Representations
Definition

Suppose that $X$ is a nonempty set (sample space) and that $\mathcal{E}$ is a nonempty family of subsets of $X$. Then $\mathcal{E}$ is an algebra of sets on $X$ iff, for every $A, B \in \mathcal{E}$:

1. $-A \in \mathcal{E}$.
2. $A \cup B \in \mathcal{E}$.
Definition

Suppose that $X$ is a nonempty set (sample space) and that $\mathcal{E}$ is a nonempty family of subsets of $X$. Then $\mathcal{E}$ is an algebra of sets on $X$ iff, for every $A, B \in \mathcal{E}$:

1. $-A \in \mathcal{E}$.
2. $A \cup B \in \mathcal{E}$

Furthermore, if $\mathcal{E}$ is closed under countable unions, the $\mathcal{E}$ is called a $\sigma$-algebra on $X$. 
Kolmogorov Axioms

Definition

Suppose that $X$ is a nonempty set, that that $\mathcal{E}$ is an algebra of sets on $X$, and that $P$ is a function from $\mathcal{E}$ into the real numbers. The triple $\langle X, \mathcal{E}, P \rangle$ is a (finitely additive) probability space iff, for every $A, B \in \mathcal{E}$:

1. $P(A) \geq 0$.
2. $P(X) = 1$.
3. If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$.
Kolmogorov Axioms

**Definition**

It is a probability space \( \langle X, \mathcal{E}, P \rangle \) is **countably additive** if in addition:

1. \( \mathcal{E} \) is a \( \sigma \)-algebra on \( X \).
2. If \( A_i \in \mathcal{E} \) and \( A_i \cap A_j = \emptyset, i \neq j \), then

\[
P \left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} P(A_i).\]
finite $X$ + algebra $\Rightarrow \sigma$-algebra

finite $X$ + probability space $\Rightarrow$ countably additive probability space

$\langle X, \mathcal{E}, P \rangle$ measure space + $P(X) = 1$ $\iff \langle X, \mathcal{E}, P \rangle$ countably additive probability space

Non-countably-additive probability space $\iff$ infinite $X$ + (non-$\sigma$) algebra
Necessary Conditions

**Definition**

Suppose that $X$ is a nonempty set, that $\mathcal{E}$ is an algebra of sets on $X$, and that $\succsim$ is a relation on $\mathcal{E}$. The triple $\langle X, \mathcal{E}, \succsim \rangle$ is a **structure of qualitative probability** iff for every $A, B, C \in \mathcal{E}$:

1. $\langle \mathcal{E}, \succsim \rangle$ is a weak ordering.
2. $X \succsim \emptyset$ and $A \succsim \emptyset$.
3. Suppose that $A \cap B = A \cap C = \emptyset$. Then $B \succsim C$ iff $A \cup B \succsim A \cup C$. 
Necessary Conditions

Definition

Suppose $\mathcal{E}$ is an algebra of sets an that $\sim$ is an equivalence relation on $\mathcal{E}$. A sequence $A_1, \ldots, A_i, \ldots$, where $A_i \in \mathcal{E}$, is a standard sequence relative to $A \in \mathcal{E}$ iff there exist $B_i, C_i \in \mathcal{E}$ such that:

(i) $A_1 = B_1$ and $B_1 \sim A$;
(ii) $B_i \cap C_i = \emptyset$;
(iii) $B_i \sim A_i$;
(iv) $C_i \sim A$;
(v) $A_{i+1} = B_i \cup C_i$. 

A structure of qualitative probability is **Archimedean** iff, for every $A \succ \emptyset$, any standard sequence relative to $A$ is finite.
Nonsufficiency of Qualitative Probability

Let $X = \{a, b, c, d, e\}$ and let $\mathcal{E}$ be all subsets of $X$. Consider any order for which

\[(1)\] \{a\} \succ \{b, c\}, \quad \{c, d\} \succ \{a, b\} \quad \text{and} \quad \{b, e\} \succ \{a, c\}.

Let \( X = \{a, b, c, d, e\} \) and let \( \mathcal{E} \) be all subsets of \( X \). Consider any order for which

\[
\text{(1) } \{a\} \succ \{b, c\}, \quad \{c, d\} \succ \{a, b\} \quad \text{and} \quad \{b, e\} \succ \{a, c\}. 
\]

**Proposition**

If the relation \( \succ \) on \( \mathcal{E} \) satisfies (1) and has an order-preserving (finitely additive) probability representation, then

\[
\{d, e\} \succ \{a, b, c\}. 
\]

**Proposition**

There is a relation \( \succ \) such that \( \langle X, \mathcal{E}, \succ \rangle \) is a structure of qualitative probability and \( \{a, b, c\} \succ \{d, e\} \).
A probability representation has **metrical structure** that a (Archimedean) structure of qualitative probability does not.
Lesson?

A probability representation has **metrical structure** that a (Archimedean) structure of qualitative probability does not.

Recall that, to solve this sort of problem wrt extensive measurement, we had axiom (4) in Definition 3 of Chapter 3 (p. 84). Why not impose a similar axiom here?
Lesson?

A probability representation has **metrical structure** that a (Archimedean) structure of qualitative probability does not.

Recall that, to solve this sort of problem wrt extensive measurement, we had axiom (4) in Definition 3 of Chapter 3 (p. 84). Why not impose a similar axiom here?

*What a great idea! Let’s call it ‘Axiom 5’.***
Sufficient Conditions

Axiom 5

Suppose \( \langle X, \mathcal{E}, \succsim \rangle \) is a structure of qualitative probability. If \( A, B, C, D \in \mathcal{E} \) are such that \( A \cap B = \emptyset \), \( A \succ C \), and \( B \succsim D \), then there exist \( C', D', E \in \mathcal{E} \) such that:

(i) \( E \sim A \cup B \);

(ii) \( C' \cap D' = \emptyset \);

(iii) \( E \supset C' \cup D' \);

(iv) \( C' \sim C \) and \( D' \sim D \).
Proposition

If a finite structure of qualitative probability satisfies Axiom 5, then its equivalence classes form a single standard sequence.
Sufficient Condition

Proposition
If a finite structure of qualitative probability satisfies Axiom 5, then its equivalence classes form a single standard sequence.

Similar to “Lego blocks” in the case of extensive measurement.
Theorem 2

Suppose that \( \langle X, \mathcal{E}, \succsim \rangle \) is an Archimedean structure of qualitative probability for which Axiom 5 holds, then there exists a unique order-preserving function \( P \) such that \( \langle X, \mathcal{E}, P \rangle \) is a finitely additive probability space.
Countably Additive Representation

Definition

Suppose that $\langle X, E, \succsim \rangle$ is a structure of qualitative probability and that $E$ is a $\sigma$-algebra. We say that $\succsim$ is monotonically continuous on $E$ iff for any sequence $A_1, A_2, \ldots$ in $E$ and any $B \in E$, if $A_i \subseteq A_{i+1}$ and $B \succsim A_i$, for all $i$, then $B \succsim \bigcup_{i=1}^{\infty} A_i$.

Theorem 4

A finitely additive probability representation of a structure of qualitative probability, on a $\sigma$-algebra, is countably additive iff the structure is monotonically continuous.
Countably Additive Representation

Definition

Suppose that $\langle X, \mathcal{E}, \succsim \rangle$ is a structure of qualitative probability and that $\mathcal{E}$ is a $\sigma$-algebra. We say that $\succsim$ is monotonically continuous on $\mathcal{E}$ iff for any sequence $A_1, A_2, \ldots$ in $\mathcal{E}$ and any $B \in \mathcal{E}$, if $A_i \subset A_{i+1}$ and $B \succsim A_i$, for all $i$, then $B \succsim \bigcup_{i=1}^{\infty} A_i$.
Countably Additive Representation

**Definition**

Suppose that \( \langle X, \mathcal{E}, \succ \rangle \) is a structure of qualitative probability and that \( \mathcal{E} \) is a \( \sigma \)-algebra. We say that \( \succ \) is monotonically continuous on \( \mathcal{E} \) iff for any sequence \( A_1, A_2, \ldots \) in \( \mathcal{E} \) and any \( B \in \mathcal{E} \), if \( A_i \subset A_{i+1} \) and \( B \succ A_i \), for all \( i \), then \( B \succ \bigcup_{i=1}^{\infty} A_i \).

**Theorem 4**

A finitely additive probability representation of a structure of qualitative probability, on a \( \sigma \)-algebra, is countably additive iff the structure is monotonically continuous.
Countably Additive Representation

Definition

Let $\succsim$ be a weak ordering of an algebra of sets $\mathcal{E}$. An even $A \in \mathcal{E}$ is an atom iff $A \succsim \mathcal{E}$ and for any $B \in \mathcal{E}$, if $A \supset B$, then $A \sim B$ or $B \sim \emptyset$. 

Countably Additive Representation

**Definition**

Let $\succsim$ be a weak ordering of an algebra of sets $\mathcal{E}$. An even $A \in \mathcal{E}$ is an atom iff $A \succsim \mathcal{E}$ and for any $B \in \mathcal{E}$, if $A \supset B$, then $A \sim B$ or $B \sim \emptyset$.

**Theorem 5**

Suppose that $\langle X, \mathcal{E}, \succsim \rangle$ is a structure of qualitative probability, $\mathcal{E}$ is a $\sigma$-algebra, and there are no atoms. Then there is a unique order preserving probability representation, and it is countably additive.
QM-Algebra

Suppose that $X$ is a nonempty set and that $E$ is a nonempty family of subsets of $X$. Then $E$ is a QM-algebra of sets on $X$ iff,

1. $-A \in E$;
2. If $A \cap B = \emptyset$, then $A \cup B \in E$.

Furthermore, if $E$ is closed under countable unions of mutually disjoint sets, then $E$ is called a QM $\sigma$-algebra.
QM-Algebra

Definition

Suppose that $X$ is a nonempty set and that $\mathcal{E}$ is a nonempty family of subsets of $X$. Then $\mathcal{E}$ is a QM-algebra of sets on $X$ iff, for every $A, B \in \mathcal{E}$

1. $-A \in \mathcal{E}$;
2. If $A \cap B = \emptyset$, then $A \cup B \in \mathcal{E}$.

Furthermore, if $\mathcal{E}$ is closed under countable unions of mutually disjoint sets, then $\mathcal{E}$ is called a QM $\sigma$-algebra.
Axiom 3′

Suppose that $A \cap B = C \cap D = \emptyset$. If $A \succ C$ and $B \succ D$, then $A \cup B \succ C \cup D$; moreover, if either hypothesis is $\prec$, then the conclusion is $\succ$. 
Axiom 3′
Suppose that $A \cap B = C \cap D = \emptyset$. If $A \succ C$ and $B \succ D$, then $A \cup B \succ C \cup D$; moreover, if either hypothesis is $\succ$, then the conclusion is $\succ$.

Theorem 3
If $\mathcal{E}$ is a QM-algebra and if $\langle X, \mathcal{E}, \succ \rangle$ satisfies Axioms 1, 2, 3′, 4, and 5, then there is a unique order-preserving (finitely additive) probability representation on $\mathcal{E}$. 
Independent Events
Necessary Conditions

Definition
Suppose $\mathcal{E}$ is an algebra of sets on $X$ and $\perp$ is a binary relation on $\mathcal{E}$. Then $\perp$ is an independence relation iff

1. $\perp$ is symmetric.
2. For $A \in \mathcal{E}$, $\{B \mid A \perp B\} \subset \mathcal{E}$ is a QM-algebra.
Independent Events

Necessary Conditions

Definition
Suppose $\mathcal{E}$ is an algebra of sets on $X$ and $\perp$ is a binary relation on $\mathcal{E}$. Then $\perp$ is an independence relation iff

1. $\perp$ is symmetric.

2. For $A \in \mathcal{E}$, $\{B \mid A \perp B\} \subset \mathcal{E}$ is a QM-algebra.

Definition
Let $\mathcal{E}$ be an algebra of sets and $\perp$ an independence relation on $\mathcal{E}$. For $m \geq 2$, $A_1, \ldots, A_m \in \mathcal{E}$ are $\perp$-independent iff, for every $M \subset \{1, \ldots, m\}$, every $B$ in the smallest subalgebra containing $\{A_i \mid i \in M\}$, and every $C$ in the smallest subalgebra containing $\{A_i \mid i \notin M\}$, we have $B \perp C$. 
Independent Events
Necessary Conditions

Definition
Suppose that \( \langle X, \mathcal{E}, \preceq \rangle \) is a structure of qualitative probability and \( \perp \) is an independence relation on \( \mathcal{E} \). The quadruple \( \langle X, \mathcal{E}, \preceq, \perp \rangle \) is a structure of qualitative probability with independence iff

3. Suppose that \( A, B, C, D \in \mathcal{E} \), \( A \perp B \), and \( C \perp D \). If \( A \preceq C \) and \( B \succeq D \), then \( A \cap B \preceq C \cap D \); moreover, if \( A \succ C \), \( B \succ D \), and \( B \succ \emptyset \), then \( A \cap B \succ C \cap D \).
Structural Condition

**Definition**

The structure $\langle X, \mathcal{E}, \succsim, \bot \rangle$ is **complete** iff the following additional axiom holds:

4. For any $A_1, \ldots, A_m, A \in \mathcal{E}$, there exists $A' \in \mathcal{E}$ with $A' \sim A$ and $A' \perp A_i$. Moreover, if $A_1, \ldots, A_m$ are $\bot$-independent, then $A'$ can be chosen so that $A_1, \ldots, A_m, A'$ are also $\bot$-independent.
Definition

Suppose \( \langle X, \mathcal{E}, \succsim, \perp \rangle \) is a structure of qualitative probability with independence. Let \( \mathcal{N} = \{ A \mid A \sim \emptyset \} \subset \mathcal{E} \). If \( A, C \in \mathcal{E} \) and \( B, D \in \mathcal{E} - \mathcal{N} \), define

\[
A \mid B \succsim' C \mid D
\]

iff there exist \( A', B', C', D' \in \mathcal{E} \) with

\[
A' \sim A \cap B, \quad B' \sim B, \quad C' \sim C \cap D, \quad D' \sim D ;
\]

\[
A' \perp D' \quad \text{and} \quad C' \perp B' ;
\]

and

\[
A' \cap D' \succsim C' \cap B'.
\]
Conditional Probability

Definition

The structure $\langle X, E, \succsim, \bot \rangle$ is Archimedean iff every standard sequence is finite, where $\{A_i\}$ is a standard sequence iff for all $i$, $A_i \in E - \mathcal{N}$, $A_{i+1} \supset A_i$, and

\[ X \succsim X \succsim A_i | A_{i+1} \sim A_1 | A_2. \]
Axiom 8

If $A|B \preceq' C|D$, then there exists $C' \in \mathcal{E}$ such that $C \cap D \subset C'$ and $A|B \sim' C'|D$. 
Conditional Probability

Axiom 8

If $A|B \simeq^* C|D$, then there exists $C' \in \mathcal{E}$ such that $C \cap D \subset C'$ and $A|B \sim^* C'|D$.

* Axiom 8 is somewhere in strength between Axiom 5 and Axiom 5'. In particular, it requires an infinite sample space.
Theorem 7

Suppose that \( \langle X, \mathcal{E}, \succsim, \bot \rangle \) is an Archimedean and complete structure of qualitative probability with independence such that Axiom 8 is satisfied. Then there is a unique probability representation in which conditional probabilities preserve \( \succsim' \).
Conditional Probability

Theorem 7

Suppose that $\langle X, \mathcal{E}, \succsim, \perp \rangle$ is an Archimedean and complete structure of qualitative probability with independence such that Axiom 8 is satisfied. Then there is a unique probability representation in which conditional probabilities preserve $\succsim'$.

* Axiom 8 is somewhere in strength between Axiom 5 and Axiom 5'. In particular, it requires an infinite sample space.
Chapter 6:
Additive Conjoint Measurement
Decomposable Structures

Definition

Let $A_1, A_2$ be nonempty sets, and let $\preceq$ be a weak ordering on $A_1 \times A_2$. The triple $\langle A_1, A_2, \preceq \rangle$ is decomposable if there are real valued functions $\phi_1 : A_1 \to \mathbb{R}$, $\phi_2 : A_2 \to \mathbb{R}$, and $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, where $F$ is 1-1 in each variable, such that, for all $a, b \in A_1$ and $p, q \in A_2$,

$$ap \preceq bq \text{ iff } F[\phi_1(a), \phi_2(p)] \geq F[\phi_1(b), \phi_2(q)].$$
A decomposable structure $\langle A_1, A_2, \succsim \rangle$ is additively independent if, for all $a, b \in A_1$ and $p, q \in A_2$,

$$ap \succsim bq \text{ iff } \phi_1(a) + \phi_2(p) \geq \phi_1(b) + \phi_2(q).$$
Examples

Proposition
Suppose \( \langle A_1, A_2, \succeq \rangle \) is a decomposable structure such that

\[
ap \succeq bq \quad \text{iff} \quad \psi_1(a)\psi_2(p) \geq \psi_1(b)\psi_2(q),
\]

for positive real-valued functions \( \psi_1, \psi_2 \). Then \( \langle A_1, A_2, \succeq \rangle \) is additively independent.
Examples

Proposition

Suppose $\langle A_1, A_2, \succ \rangle$ is a decomposable structure such that

$$ ap \succ bq \iff \psi_1(a)\psi_2(p) \geq \psi_1(b)\psi_2(q), $$

for positive real-valued functions $\psi_1, \psi_2$. Then $\langle A_1, A_2, \succ \rangle$ is additively independent.

$$ ap \succ bq \iff \log \psi_1(a) + \log \psi_2(p) \geq \log \psi_1(b) + \log \psi_2(q) $$
Examples

**Momentum**

\[ p = m v \]

\[ m_1 v_1 \geq m_2 v_2 \quad \text{iff} \quad \log m_1 + \log v_1 \geq \log m_2 + \log v_2 \]
Independent Random Variables

Suppose $Y_1, Y_2$ are random variables on the same probability space, and let $\sigma(Y_i)$ be the smallest $\sigma$-algebra for which $Y_i$ is continuous. Define $\succeq$ on $\sigma(Y_1) \times \sigma(Y_2)$ by

$$ap \succeq bq \text{ iff } \Pr(a \cap p) \geq \Pr(b \cap q),$$

for all $a, b \in \sigma(Y_1)$ and $p, q \in \sigma(Y_2)$. 

Proposition $\langle \sigma(Y_1), \sigma(Y_2), \succeq \rangle$ is additively independent if and only if $X_1$ and $X_2$ are independent.
Examples

Independent Random Variables
Suppose $Y_1$, $Y_2$ are random variables on the same probability space, and let $\sigma(Y_i)$ be the smallest $\sigma$-algebra for which $Y_i$ is continuous. Define $\succsim$ on $\sigma(Y_1) \times \sigma(Y_2)$ by

$$ap \succsim bq \iff Pr(a \cap p) \geq Pr(b \cap q),$$

for all $a, b \in \sigma(Y_1)$ and $p, q \in \sigma(Y_2)$.

Proposition
$\langle \sigma(Y_1), \sigma(Y_2), \succsim \rangle$ is additively independent if and only if $X_1$ and $X_2$ are independent.
Examples

Expected Utility

Suppose $\langle X, \mathcal{E}, Pr \rangle$ is a probability space and $\mathcal{A}$ is a set of commodities with associated utility function $U$. Define $\succeq$ on $\mathcal{E} \times \mathcal{A}$ by

$$ap \succeq bq \iff Pr(a)U(p) \geq Pr(b)U(q),$$

for all $a, b \in \mathcal{E}$ and $p, q \in \mathcal{A}$. 
Definition

A relation $\succeq$ on $A_1 \times A_2$ is independent iff, for all $a, b \in A_1$, $ap \succeq bp$ for some $p \in A_2$ implies that $aq \succeq bq$ for every $q \in A_2$; and, for all $p, q \in A_2$, $ap \succeq aq$ for some $a \in A_1$ implies that $bq \succeq bp$ for every $b \in A_1$. 

* $\succeq$ is an independent relation if $\langle A_1, A_2, \succeq \rangle$ is additively independent.
Necessary Conditions
Independence (a.k.a. single cancelation)

**Definition**

A relation $\prec$ on $A_1 \times A_2$ is independent iff, for all $a, b \in A_1$, $ap \prec bp$ for some $p \in A_2$ implies that $aq \prec bq$ for every $q \in A_2$; and, for all $p, q \in A_2$, $ap \prec aq$ for some $a \in A_1$ implies that $bq \prec bp$ for every $b \in A_1$.

* $\prec$ is an independent relation if $\langle A_1, A_2, \prec \rangle$ is additively independent.
Necessary Conditions
Independence (a.k.a. single cancelation)

Definition
Suppose that \( \preceq \) is an independent relation on \( A_1 \times A_2 \).

(i) Define \( \preceq_1 \) on \( A_1 \): for \( a, b \in A_1 \), \( a \preceq_1 b \) iff \( ap \preceq bp \) for some \( p \in A_2 \); and

(ii) define \( \preceq_2 \) on \( A_2 \) similarly.
Necessary Conditions
Independence (a.k.a. single cancelation)

Lemma 1
If $\succsim$ is an independent weak ordering of $A_1 \times A_2$, then

(i) $\succsim_i$ is a weak ordering of $A_i$.
(ii) For $a, b \in A_1$ and $p, q \in A_2$, if $a \succsim_1 b$ and $p \succsim_2 q$, then $ap \succsim bq$.
(iii) If either antecedent inequality of (ii) is strict, so is the conclusion.
(iv) For $a, b \in A_1$ and $p, q \in A_2$, if $ap \sim bq$, then $a \succsim_1 b$ iff $q \succsim_2 p$. 
A relation $\succeq$ on $A_1 \times A_2$ satisfies **double cancelation** provided that, for every $a, b, f \in A_1$ and $p, q, x \in A_2$, if $ax \succeq fq$ and $fp \succeq bx$, then $ap \succeq bq$. The weaker condition in which $\succeq$ is replaced by $\sim$ is the **Thomsen condition**.
Necessary Conditions
Archimedean Axiom

Definition
Suppose \( \succeq \) is an independent weak ordering of \( A_1 \times A_2 \). For any set \( N \) of consecutive integers (positive or negative, finite or infinite), a set \( \{ a_i \mid a_i \in A_1, i \in N \} \) is a standard sequence on component 1 iff there exists \( p, q \in A_2 \) such that not(\( p \sim_2 q \)) and, for all \( i, i + 1 \in N \), \( a_ip \sim a_{i+1}q \). A parallel definition holds for the second component.
Necessary Conditions

Archimedean Axiom

Definition

Suppose $\succ$ is an independent weak ordering of $A_1 \times A_2$. For any set $N$ of consecutive integers (positive or negative, finite or infinite), a set $\{a_i \mid a_i \in A_1, i \in N\}$ is a standard sequence on component 1 iff there exists $p, q \in A_2$ such that not$(p \sim_2 q)$ and, for all $i, i+1 \in N$, $a_ip \sim a_{i+1}q$. A parallel definition holds for the second component.

Definition

A standard sequence on component 1 $\{a_i \mid i \in N\}$ is strictly bounded iff there exist $a, b \in A_2$ such that, for all $i \in N$, $c \succ_1 a_i \succ_1 b$. A parallel definition holds for the second component.
necessary conditions

archimedean axiom

definition

Suppose \( \succsim \) is an independent weak ordering of \( A_1 \times A_2 \). \( \langle A_1, A_2, \succsim \rangle \) is Archimedean iff every strictly bounded standard sequence (on either component) is finite.
A relation \( \succsim \) on \( A_1 \times A_2 \) satisfies **unrestricted solvability** provided that, given three of \( a, b \in A_1 \) and \( p, q \in A_2 \), the fourth exists so that \( ap \sim bq \).
Sufficient Condition

Solvability

Definition

A relation \( \succcurlyeq \) on \( A_1 \times A_2 \) satisfies restricted solvability provided that:

(i) whenever there exist \( a, \bar{b}, \underline{b} \in A_1 \) and \( p, q \in A_2 \) for which \( \bar{b}q \succcurlyeq ap \succcurlyeq bq \), then there exists \( b \in A_1 \) such that \( bq \sim ap \);

(ii) a similar condition holds on the second component.
Definition

Suppose that $\succsim$ is a relation on $A_1 \times A_2$. Component $A_1$ is **essential** iff there exist $a, b \in A_1$ and $p \in A_2$ such that not($ap \sim bp$). A similar definition holds for $A_2$. 
Definition
Suppose that $\sim$ is a relation on $A_1 \times A_2$. Component $A_1$ is essential iff there exist $a, b \in A_1$ and $p \in A_2$ such that not($ap \sim bp$). A similar definition holds for $A_2$.

Lemma 2
Suppose that $\succsim$ is an independent relation on $A_1 \times A_2$. Then component $A_1$ is essential iff there exist $a, b \in A_1$ such that $a \succsim_1 b$. 
Additive Conjoint Structure

Definition

Suppose that $A_1$ and $A_2$ are nonempty sets and $\succeq$ is a binary relation on $A_1 \times A_2$. The triple $\langle A_1, A_2, \succeq \rangle$ is an additive conjoint structure iff $\succeq$ satisfies the following six axioms:

1. Weak ordering
2. Independence
3. Thomsen condition
4. Restricted solvability
5. Archemedean property
6. Each component is essential

The structure is symmetric iff, in addition,

7. For $a, b \in A_1$, there exist $p, q \in A_2$ such that $ap \sim bq$, and for $p', q' \in A_2$, there exist $a', b' \in A_1$ such that $a'p' \sim b'q'$. 
Theorem 1
Suppose $\langle A_1, A_2, \succsim \rangle$ is a structure for which the weak ordering, double cancellation, unrestricted solvability, and the Archimedean axioms hold. If at least one component is essential, then $\langle A_1, A_2, \succsim \rangle$ is a symmetric, additive conjoint structure.
Theorem 2

Suppose $\langle A_1, A_2, \succsim \rangle$ is an additive conjoint structure. Then there exist functions $\phi_i : A_i \to \mathbb{R}$ such that, for all $a, b \in A_1$ and $p, q \in A_2$,

$$ap \succsim bq \text{ iff } \phi_1(a) + \phi_2(p) \geq \phi_1(b) + \phi_2(q).$$

If $\phi'_i$ are two other functions with the same property, then there exists constants $\alpha > 0, \beta_1$ and $\beta_2$ such that

$$\phi'_1 = \alpha \phi_1 + \beta_1 \quad \text{and} \quad \phi'_2 = \alpha \phi_2 + \beta_2.$$
Representation Theorem
Uniqueness of multiplicative representation

Proposition
Suppose \( \langle A_1, A_2, \succeq \rangle \) is an additive conjoint structure. Then there exist functions \( \psi_i : A_i \to \mathbb{R}^+ \) such that, for all \( a, b \in A_1 \) and \( p, q \in A_2 \),

\[
ap \preceq bq \quad \text{iff} \quad \psi_1(a)\psi_2(p) \geq \psi_1(b)\psi_2(q).
\]

If \( \psi'_i \) are two other functions with the same property, then there exists constants \( \alpha > 0, \beta_1 \) and \( \beta_2 \) such that

\[
\phi'_1 = \beta_1 \psi_1^\alpha \quad \text{and} \quad \psi'_2 = \beta_2 \psi_2^\alpha.
\]
Suppose $\langle A_1, A_2, \succcurlyeq \rangle$ is a symmetric, additive conjoint structure. It is bounded iff there are $\bar{a}, \tilde{a} \in A_1$, $\bar{p}, \tilde{p} \in A_2$ such that

$$\bar{a} \bar{p} \sim \tilde{a} \tilde{p}$$

and, for $a \in A_1$ and $p \in A_2$,

$$\tilde{a} \succcurlyeq_1 a \succcurlyeq_1 a \text{ and } \tilde{p} \succcurlyeq_2 p \succcurlyeq_2 p.$$
Moreover, for \(a, b \in A_1\), we define: 
\[
\pi(a) \in A_2
\]
as the (unique up to \(\sim_2\)) solution to 
\[
a \pi(a) \sim a \pi(b);
\]
\[
B_1 = \\{ab | a, b \succ_1 a \text{ and } \bar{a} \pi \succeq a \pi(b)\};
\]
for \(ab \in B_1\), 
\[
a \circ b
\]
is the (unique up to \(\sim_1\)) solution to 
\[
(a \circ b) \sim a \pi(b).
\]
Similar definitions hold for \(A_2\) with \(\alpha(p)\) playing the role of \(\pi(a)\).

**Lemma 5**

If \(\langle A_1, A_2, \succeq \rangle\) is a bounded, symmetric, additive conjoint structure, and if \(B_1\) is nonempty, then \(\langle A_1, \succeq_1, B_1, \circ \rangle\) is an extensive structure with no essential maximum.
Moreover, for \(a, b \in A_1\), we define: \(\pi(a) \in A_2\) as the (unique up to \(\sim_2\)) solution to \(a\pi(a) \sim ap\); \(B_1 = \{ab \mid a, b \succ_1 a \text{ and } \bar{ap} \succ \bar{a}\pi(b)\}\); for \(ab \in B_1\), \(a \circ b\) is the (unique up to \(\sim_1\)) solution to \((a \circ b)p \sim a\pi(b)\). Similar definitions hold for \(A_2\) with \(\alpha(p)\) playing the role of \(\pi(a)\).

**Lemma 5**

If \(\langle A_1, A_2, \succ \rangle\) is a bounded, symmetric, additive conjoint structure, and if \(B_1\) is nonempty, then \(\langle A_1, \succ_1, B_1, \circ \rangle\) is an extensive structure with no essential maximum.
Define the dual relations \( \preceq^{'} \) and \( \preceq^{''} \) as follows:

\[
ap \preceq^{'} b \iff aq \preceq^{''} bp
\]

Theorem 5

If two relations are dual, then transitivity and double cancellation are dual properties, and independence, restricted and unrestricted solvability, and the Archemedean property are self-dual properties.
Define the dual relations $\preceq'$ and $\preceq'$ as follows:

$$ap \preceq bq \text{ iff } aq \preceq' bp.$$  

**Theorem 5**

If two relations are dual, then transitivity and double cancellation are dual properties, and independence, restricted and unrestricted solvability, and the Archimedean property are self-dual properties.