Space and Time*

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Abstract

Here, formal tools are used to pose and answer several philosophical questions concerning space and time. The questions involve the properties of possible worlds allowed by the general theory of relativity. In particular, attention is given to various causal properties such as "determinism" and "time travel".

Keywords: Space, Time, Determinism, Time Travel, General Relativity

1 Introduction

It is no surprise that formal methods have proven to be quite useful in the philosophy of space and time. With them, great progress has been made on the question, heavily debated since Newton and Leibniz, of whether space and time are absolute or relational in character. And there is a related problem which has also been clarified considerably: whether or not various geometrical properties and relations are matters of convention. [Sklar 1976; Friedman 1983; Earman 1989]

These topics, interesting as they are, will not be considered here. Rather, the focus will concern the "global structure" of space and time. General relativity (our best large-scale physical theory) will be presupposed. But the investigation of global structure will allow us to step away from the complex details of this theory and instead examine space and time with an eye towards a number of fundamental features (e.g. topology, causal structure). [Geroch and Horowitz 1979]

An elegant mathematical formalism is central to the subject. So too are the associated space-time diagrams. Using these tools, questions of physical and philosophical interest can be posed and answered. A small subset of these

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questions are examined below. A number of others are discussed elsewhere. [Earman 1995]

2 Possible Worlds

General relativity determines a class of cosmological models. Each model represents a physically possible world which is compatible with the theory. We take such a model (also called a *spacetime*) to be an ordered pair (M, g). Here, M is a connected smooth, n-dimensional manifold ($n \geq 2$) and g is a smooth, Lorentzian metric on M. (Usually n is taken to be four, but possible worlds with other dimensions are also considered. For ease of presentation, a number of two-dimensional models will be examined here.)

The manifold M captures the shape (topology) of the universe and each point in M represents a possible event. From our experience, it seems that any event (a first kiss, for example) can be characterized by n numbers (one temporal and $n-1$ spatial coordinates). So naturally, the local structure of M is characterized by an n-dimensional Cartesian coordinate system. But globally, M need not have the same structure. Indeed, M can have a variety of possible shapes. A number of two-dimensional manifolds are familiar to us: the plane, the sphere, the cylinder, the torus. Note too that any manifold with any closed set of points removed also counts as a manifold. For example, the sphere with the "North pole" removed is a manifold.

A manifold does a fine job of representing the totality of possible events but more structure is needed to capture exactly how these possible events are related. The Lorentzian metric g provides this extra structure. We can think of g as a type of function which assigns a length to any vector at any point in M. But it is crucial that, at every point in M, the metric g not only assign some positive lengths but also some zero and negative lengths as well. In this way, g partitions all vectors at a point in M into three non-empty classes: the timelike (positive length), the lightlike (zero length), and the spacelike (negative length). The result is a light cone structure at each point in M (see Figure 1). Physically, the light cone structure demarcates the upper bound to the velocities of massive particles (it is central to relativity theory that nothing can travel faster than light).

The light cone structure can certainly change smoothly from point to point. But it need not. In fact, a number of interesting physically possible worlds, and all of the examples considered below, have a light cone structure which remains constant (a metric with this property is said to be *flat*).

Now that we have given a characterization of physically possible worlds, we are in a position to ask a somewhat interesting question.

Question. Given any shape, is there a physically possible world with that shape? (Answer: No.)

First we translate the question into the formalism: Given any n-dimensional

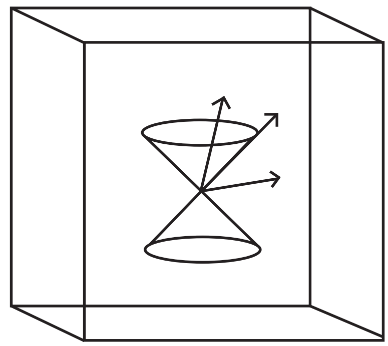

Figure 1: *The three-dimensional possible world* (M, g)*. A representative light cone is depicted. Timelike vectors are inside the light cone; spacelike vectors, outside. Lightlike vectors are on the boundary.*

manifold M, can a Lorentzian metric g be put on M? Next, we can get a grip on the question by noticing that a manifold admits a Lorentzian metric if and only if it admits a (non-vanishing) timelike vector field. [Geroch and Horowitz 1979] But an n-dimensional sphere does not admit a non-vanishing timelike vector field if n is even (this essentially follows from the famous "hairy ball theorem" of Brouwer). So, the answer to our question is negative. There is no physically possible world with an even number of dimensions (including our own) shaped like a sphere.

3 Orientability

Consider a physically possible world (M, g). The light cone which g assigns to any event in M has two lobes. And at any given event, we can certainly label one lobe as "future" and the other as "past". But can we do this for every event in M in a way that involves no discontinuities? If such a labeling is not possible, there could be no proper distinction between particles traveling "forward" and "backward" in time. If, on the other hand, such a labeling *is* possible, then we could, in a globally consistent way, give time a direction. A natural question arises here.

Question. Can time be given a direction in all physically possible worlds? (Answer: No.)

We know from the previous section that any spacetime (M, g) must admit a timelike vector field on M. The question above amounts to whether any spacetime (M, g) must admit a *continuous* timelike vector field as well. A bit of thought produces a simple counterexample: a physically possible world shaped like the Möbius strip with a flat metric (see Figure 2). In such a world, global notions of "past" and "future" are not meaningful.

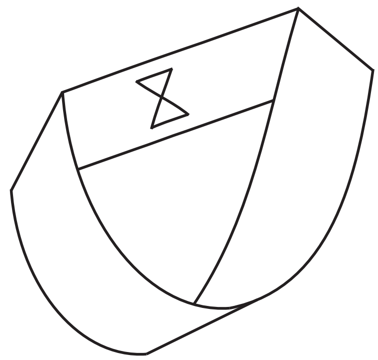

Figure 2: *The two-dimensional possible world* (M, g) *which does not admit a continuous nonvanishing timelike vector field. Here* M *is a Möbius strip.*

Let us say that a spacetime which does admit a continuous (non-vanishing) timelike vector field is *temporally orientable*. Because many other global conditions presuppose temporal orientability, it is customary to consider only spacetimes with this property. In what follows, we adhere to the custom.

4 Chronology

Suppose some physically possible world (M, g) is temporally orientable and that an orientation has been given. The next geometric object of study is the *future directed timelike curve* (sometimes called a *worldline*). It is simply a smooth curve on the manifold M such that all its tangent vectors are timelike and point to the future. A future directed timelike curve represents the possible life history of a massive particle; if there is a future directed timelike curve from some event p to some other event q, it must be, in principle, possible for a massive particle to travel from the one to the other.

(A *future directed lightlike curve* is defined analogously. A *future directed causal curve* is a smooth curve on the manifold such that all its tangent vectors are either timelike or lightlike and point to the future.)

We now are in a position to define a (two-place) relation $\ll$ on the events in M. We write $p \ll q$ if there exists a future directed timelike curve from p to q. (An analogous relation $<$ can be defined using future directed causal curves.) It is not difficult to prove that the relation $\ll$ is transitive: for any events p, q, and r, if $p \ll q$ and $q \ll r$, then $p \ll r$. At first, it also seems as though the relation cannot allow for distinct events p and q to be such that both $p \ll q$ and $q \ll p$ (which, by transitivity, would imply that $p \ll p$). In that case, a massive particle may travel from one event to another and then back again undergoing “time travel” of a certain kind. We ask the following question.

Question. Is there a physically possible world which allows for time travel? (Answer: Yes.)

Let M be a two-dimensional cylindrical manifold and let the metric g be flat and such that timelike curves are permitted to loop around the cylinder (see Figure 3). Clearly time travel is permitted since the relation $\ll$ holds between any two points in M. Due to their paradoxical time structures, physically possible worlds which allow for time travel have received a great deal of attention from philosophers. [Gödel 1949; Stein 1970] In what follows, we will say that a spacetime (M, g) satisfies the *chronology* condition if time travel is not permitted.

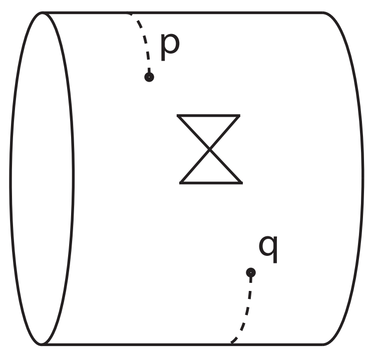

Figure 3: *The two-dimensional possible world* (M, g)*. Timelike curves are permitted to loop around the cylinder* M *so that* $p \ll q$ *for all events* p *and* q*.*

There is an interesting result concerning the shapes of physically possible worlds which satisfy the chronology condition. We say a manifold is *compact* if every sequence of its points has an accumulation point. (The sphere and torus are both compact while the plane is not.) One can show that if a spacetime (M, g) is such that M is compact, (M, g) must violate chronology. [Hawking and Ellis 1973]. However, the converse is false: Gödel spacetime is one counterexample.

5 Distinguishability

Given a physically possible world (M, g) and any event p in that world, we next can consider the collection of events in M which could have possibly influenced p. We call such a set the *past* (or *past domain of influence*) of p and formally it is defined as $I^-(p) \equiv \{q \in M : q \ll p\}$. In words, an event is a member of the past of p if there is a future directed timelike line from that event to p. Analogously, we can consider the collection of events in M which p may possibly influence. We call this set the *future* of p and define it as $I^+(p) \equiv \{q \in M : p \ll q\}$. (Analogous sets $J^-(p)$ and $J^+(p)$ can be defined using the $<$ relation.)

Are there physically possible worlds which allow distinct events to have identical pasts? Futures? There are. The example considered in the previous section (recall Figure 3) is such that for any event p, $I^-(p) = I^+(p) = M$. So, clearly, we have for any distinct events p and q, $I^-(p) = I^-(q)$ and $I^+(p) = I^+(q)$. Following standard practice, let us say that any physically possible world which allows distinct events to have identical pasts is not *past distinguishing*. Analogously, let us say that any physically possible world which allows distinct events to have identical futures is not *future distinguishing*. [Hawking and Ellis 1973]

The example in the previous section was neither past nor future distinguishing but it also did not satisfy the chronology condition. Perhaps there is some connection.

Question. If a physically possible world allows for time travel, must it allow different events to have influence over precisely the same set of future events? (Answer: Yes.)

To see the connection, assume that a spacetime (M, g) allows for time travel. So there must be distinct events p and q in M such that $p \ll q$ and $q \ll p$. From $p \ll q$, we know that $I^+(q) \subseteq I^+(p)$. From $q \ll p$, we know that $I^+(p) \subseteq I^+(q)$. Thus, $I^+(q) = I^+(p)$. So we conclude that every physically possible world which violates chronology also must violate future distinguishability. (An analogous result holds for past distinguishability.) Now, does the implication go in the other direction?

Question. Must a physically possible world allow for time travel if it allows different events to have influence over precisely the same set of future events? (Answer: No.)

A counterexample is not too hard to find. Let M be a two-dimensional cylindrical manifold and let the metric g be flat and such that only lightlike curves are permitted to loop around the cylinder (see Figure 4). This allows for the spacetime to satisfy chronology while allowing the points p and q to have the same futures (and also the same pasts).

There is a notable theorem concerning physically possible worlds which are both future and past distinguishing: Any two such worlds must have the same shape if they have the same causal structure. Formally, if (M, g) and (M', g') are past and future distinguishing and if there is a bijection $\varphi : M \to M'$ such that, for all p and q in M, $p \ll q$ if and only if $\varphi(p) \ll \varphi(q)$, then M and M' have the same topology. [Malament 1977]

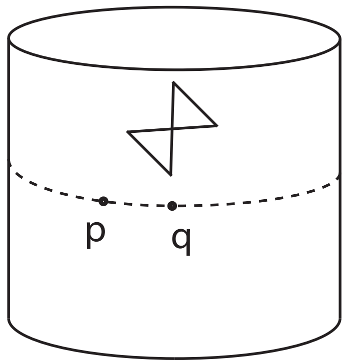

Figure 4: *The two-dimensional possible world* (M, g). *Chronology is not violated but the distinct events* p *and* q *have identical futures (the region above the dotted line) and also identical pasts (the region below the dotted line).*

6 Stability

Although the example in the previous section (recall Figure 4) did not not allow for time travel, it "almost" did. If the light cones were opened at each point, by even the slightest amount, chronology would be violated. So, there is a sense in which the example is (arbitrarily) "close" to worlds which allow for time travel. Physically possible worlds with this property are said to be not *stably causal*. Spelling out with precision the condition of stable causality requires a bit more formalism than we have available to us here. But fortunately there there is an equivalent condition which is much easier to state.

We say a spacetime (M, g) admits a *global time function* if there is a smooth function $t : M \to \mathbb{R}$ such that, for any distinct events p and q, if $p \in J^-(q)$, then $t(p) < t(q)$. It is a fundamental result that this condition which guarantees the existence of "cosmic time" is both necessary and sufficient for stable causality. [Hawking 1969] We are now in a position to investigate how stable causality is connected to past (or future) distinguishability.

Question. Is there a physically possible world which allows different events to have influence over precisely the same set of future events and yet is not close to any worlds which allow for time travel? (Answer: No.)

Although it is not immediate, a violation of future (or past) distinguishability does indeed imply a violation of stable causality. [Hawking and Ellis 1973] (The much weaker result, that a violation of chronology implies a violation of stable causality, should be clear.) Does the implication go in the other direction?

Question. Is there a physically possible world in which different events always have influence over different sets of future events and yet is close to a world which allows for time travel? (Answer: Yes.)

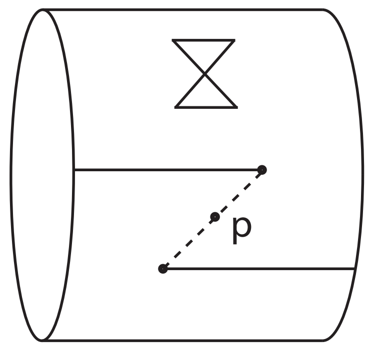

Figure 5: *The two-dimensional possible world* (M, g)*. Because of the removed strips, the future and past distinguishability conditions hold. But stable causality is violated; if the light cones were to be opened by even the slightest amount at each point, it would be the case that* $p \ll p$*.*

To construct a spacetime which satisfies future (and past) distinguishability but violates stable causality, begin with the two-dimensional cylindrical manifold and let the metric be flat and such that timelike curves are permitted to loop around the cylinder (recall Figure 3). Next, remove two strips which just prevent causal curves from connecting (see Figure 5). The result is a spacetime, call it (M, g). One can verify that for any distinct points p and q in M, $I^-(p) \neq I^-(q)$ and $I^+(p) \neq I^+(q)$. But although there *is* a function $t : M \to \mathbb{R}$ such that t increases along every future directed causal curve, no such function exists which is also smooth. So, the spacetime fails to be stably causal.

7 Determinism

What does it mean to say that a physically possibly world is deterministic? Roughly the idea is that, in such a world, all events must depend upon the events at any one time. Let us make this precise.

Consider a spacetime (M, g) and let S be any subset of M. We define the *domain of dependence* of S, $D(S)$, to be the set points p in M such that every causal curve through p, without a past or future "end point", intersects S. The set $D(S)$ represents those events in M which depend entirely upon the events in S. Next, we say that a set S is *achronal* if, for any events p and q in S, it is not the case that $p \ll q$. A set of events which are thought to be happening at any one time must certainly be achronal. Finally, we say that a spacetime has a *Cauchy surface* if there is an achronal set S in M such that $D(S) = M$. (In an n-dimensional spacetime, a Cauchy surface S necessarily has $n-1$ dimensions.)

There are a number of theorems which can be interpreted as stating that what happens on a Cauchy surface fully determines what happens throughout the entire spacetime. [Choquet-Bruhat and Geroch 1969] And although there are subtleties involved, for our purposes a physically possible world with a Cauchy surface (also called a *globally hyperbolic* spacetime) will be considered deterministic. [Earman 1986] With determinism clearly defined, one might wonder about the following.

Question. If a physically possible world is close to a world which allows for time travel, must it be indeterministic? (Answer: Yes.)

That determinism implies stable causality is non-trivial. But the result can even be strengthened: In any globally hyperbolic spacetime (M, g), a global time function $t : M \to \mathbb{R}$ can be found such that each surface of constant t is a Cauchy surface. Also, the shape of the Cauchy surfaces are all the same. [Geroch 1970] Does the implication go in the other direction?

Question. If a physically possible world is indeterministic, must it be close to another world which allows for time travel? (Answer: No.)

A counterexample is easy to construct. Let the manifold M be the two-dimensional plane with one point removed. Let the metric g be flat. The resulting spacetime (M, g) admits a global time function $t : M \to \mathbb{R}$ but for any achronal set of events S the set $D(S)$ is not M (see Figure 6). (If the point were not removed, the spacetime *would* be globally hyperbolic.) With the answer to this question, we can now note that chronology, future (or past) distinguishability, stable causality, and global hyperbolicity form a hierarchy of causal conditions (see Table 1).

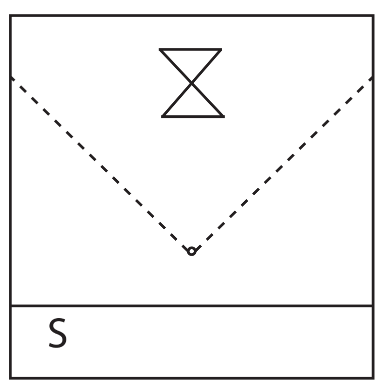

Figure 6: *The two-dimensional possible world (M, g). Stable causality is not violated but because of the removed point, any achronal surface S will be such that its domain of dependence (the region below the dotted line) is not all of M.*

Chronology	$\Leftarrow$	Future (or Past)	$\Leftarrow$	Stable	$\Leftarrow$	Global
	$\not\Rightarrow$	Distinguishability	$\not\Rightarrow$	Causality	$\not\Rightarrow$	Hyperbolicity

Table 1: Causal Hierarchy

There is a sense in which determinism is connected to the absence of "holes" in spacetime. We say a spacetime (M, g) is *internally causally compact* (i.e. it has no holes) if, for all events p and q, $J^-(p) \cap J^+(q)$ is compact. Note that a spacetime with a point removed is never internally causally compact since one can find a sequence of points without accumulation point in $J^-(p) \cap J^+(q)$ if p and q are chosen so that $J^-(p) \cap J^+(q)$ "contains" the missing point. Now one can certainly show that global hyperbolicity implies internal causal compactness. What is fascinating is that the converse is also true if future (or past) distinguishability is assumed. [Geroch 1970] Thus, putting various results together, one can understand determinism to be equivalent to the conjunction of a weak causal condition and the requirement of no holes.

8 Reasonable Worlds

The discussion so far has concerned physically possible worlds. One is also interested in a subset of these worlds: the reasonable ones. However, what counts as a physically reasonable world is not always clear and often depends upon the context. Here, we briefly mention two questions concerning physically reasonable worlds.

The early results of global structure concerned "singularities" of a certain kind. The idea was to show, using fairly conservative assumptions, that all physically reasonable worlds must necessarily contain spacetime singularities. The project culminated in a number of general theorems. [Hawking and Penrose 1970] And these theorems eventually led to serious worries concerning determinism. Indeed one natural question, still investigated today, is the following. [Penrose 1979]

Question. Is every physically reasonable world deterministic?

The question has different answers depending on how it is interpreted formally. And there is certainly much interpretive disagreement among physicists and philosophers. [Earman 1995] If we can agree, for the time being, that not every physically reasonable spacetime is deterministic, then there is another question of interest.

Question. Is there a physically reasonable world which allows for time travel?

Under some formal interpretations, the question has a negative answer.

[Hawking 1992] Under others, the question is still open. [Earman et al. 2009] As before, the entire debate hinges on the details concerning the meaning of a "physically reasonable" world. [Manchak 2013] And of course, such details are best articulated and explored with the formalism.

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