On the Existence of "Time Machines" in General Relativity

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Within the context of general relativity, we consider one definition of a "time machine" proposed by Earman, Smeenk, and Wüthrich. They conjecture that, under their definition, the class of time machine spacetimes is not empty. Here, we prove this conjecture.

1. Introduction. One peculiar feature of general relativity concerns the existence of closed timelike curves in some cosmological models permitted by the theory. In such models, a massive point particle may both commence and conclude a journey through spacetime at one and the same point. In this respect, these models allow for "time travel."

Naturally, the existence of closed timelike curves in some relativistic models prompts fascinating questions.[1] One issue, recently addressed by Earman, Smeenk, and Wüthrich (2009), concerns what it might mean to say that a model allows for the operation of a "time machine" in some sense.[2] They propose a precise definition and then conjecture that, under their formulation, there exist cosmological models that count as time machines. In this article, we provide a proof of this conjecture.

2. Background Structure. We begin with a few preliminaries concerning the relevant background formalism of general relativity.[3] An n-dimensional, relativistic *spacetime* (for $n \geq 2$) is a pair of mathematical objects (M, g_{ab}),

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1. For a thorough investigation of many of these questions, see Earman 1995.

2. See also Earman and Wüthrich 2004.

3. The reader is encouraged to consult Hawking and Ellis 1973 and Wald 1984 for details. An outstanding (and less technical) survey of the global structure of spacetime is given by Geroch and Horowitz 1979.

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where M is a connected n-dimensional manifold (without boundary) that is smooth (infinitely differentiable) and g_{ab} is a smooth, nondegenerate, pseudo-Riemannian metric of Lorentz signature $(+, -, \ldots, -)$ defined on M. Each point in the manifold M represents an "event" in spacetime.

For each point $p \in M$, the metric assigns a cone structure to the tangent space M_p. Any tangent vector ξ^a in M_p will be *timelike* (if $g_{ab}\xi^a\xi^b > 0$), *null* (if $g_{ab}\xi^a\xi^b = 0$), or *spacelike* (if $g_{ab}\xi^a\xi^b < 0$). Null vectors create the cone structure; timelike vectors are inside the cone, while spacelike vectors are outside. A *time-orientable* spacetime is one that has a continuous timelike vector field on M. A time-orientable spacetime allows us to distinguish between the future and past lobes of the light cone. In what follows, it is assumed that spacetimes are time orientable.

For some interval $I \subseteq \mathbb{R}$, a smooth curve $\gamma : I \to M$ is *timelike* if the tangent vector ξ^a at each point in $\gamma[I]$ is timelike. Similarly, a curve is *null* (respectively, *spacelike*) if its tangent vector at each point is null (respectively, spacelike). A curve is *causal* if its tangent vector at each point is either null or timelike. A causal curve is *future directed* if its tangent vector at each point falls in or on the future lobe of the light cone. Given a point $p \in M$, the *causal future of p* (written $J^+(p)$) is the set of points $q \in M$ such that there exists a future-directed causal curve from p to q. Naturally, for any set $S \subseteq M$, define $J^+[S]$ to be the set $\cup\{J^+(x) : x \in S\}$. A *chronology-violating region* $V \subseteq M$ is the set of points $p \in M$ such that there is a closed timelike curve through p.

A point $p \in M$ is a *future endpoint* of a future-directed causal curve $\gamma : I \to M$ if, for every neighborhood O of p, there exists a point $t_0 \in I$ such that $\lambda(t) \in O$ for all $t > t_0$. A *past endpoint* is defined similarly. For any set $S \subseteq M$, we define the *past domain of dependence of S* (written $D^-(S)$) to be the set of points $p \in M$ such that every causal curve with past endpoint p and no future endpoint intersects S. The *future domain of dependence of S* (written $D^+(S)$) is defined analogously. The entire *domain of dependence of S* (written $D(S)$) is just the set $D^-(S) \cup D^+(S)$.

A set $S \subset M$ is *achronal* if no two points in S can be connected by a timelike curve. A set $S \subset M$ is a *slice* if it is closed, achronal, and without edge. A set $S \subset M$ is a *spacelike surface* if S is an $(n-1)$–dimensional submanifold (possibly with boundary) such that every curve in S is spacelike.[4]

Two spacetimes (M, g_{ab}) and (M', g'_{ab}) are *isometric* if there is a diffeomorphism $\phi : M \to M'$ such that $\phi_*(g_{ab}) = g'_{ab}$. We say that a spacetime (M', g'_{ab}) is a (proper) *extension* of (M, g_{ab}) if there is a proper subset N

4. Allowing S to have a boundary is nonstandard, but the formulation introduces no difficulties. In particular, one may consider initial data on S and determine its domain of dependence $D(S)$. See Hawking and Ellis 1973, 201.

of M' such that (M, g_{ab}) and $(N, g'_{ab|N})$ are isometric. We say a spacetime is *inextendible* if it has no proper extension.

3. A Time Machine. In their recent (2009) paper, Earman et al. attempt to clarify what it might mean to say that a time machine operates within a relativistic spacetime. First, in order to count as a time machine, a spacetime (M, g_{ab}) must contain a spacelike slice S representing a "time" before the time machine is switched on. Next, they note that a time machine should operate within a finite region of spacetime. Accordingly, they require that the time machine region $T \subset M$ have compact closure. In addition, so as to guarantee that instructions for the operation of the time machine (set on S) are followed, they require that $T \subset D^+(S)$. Of course, the spacetime (M, g_{ab}) must also have a chronology-violating region V to the causal future of the time machine region T.

Finally, in order to capture the idea that a time machine must "produce" closed timelike curves, Earman et al. demand that every suitable extension of $D(S)$ contain a chronology-violating region V'. For them, a suitable extension must be inextendible and satisfy a condition known as "hole freeness." This condition, introduced by Geroch (1977), essentially requires that the domain of dependence $D(\Sigma)$ of each spacelike surface Σ be "as large as it can be." Here, hole freeness serves to rule out extensions of $D(S)$ that fail to have closed timelike curves only because of the formation of seemingly artificial "holes" in spacetime.[5] Formally, we say a spacetime (M, g_{ab}) is *hole free* if, for any spacelike surface Σ in M, there is no isometric embedding $\theta : D(\Sigma) \to M'$ into another spacetime (M', g'_{ab}) such that $\theta(D(\Sigma)) \neq D(\theta(\Sigma))$. We can now state the definition of a time machine.

> **Definition.** A spacetime (M, g_{ab}) is an *ESW time machine* if (i) there is a spacelike slice $S \subset M$, a set $T \subset M$ with compact closure, and a chronology-violating region $V \subset M$ such that $T \subset D^+(S)$ and $V \subset J^+[T]$ and (ii) every hole-free, inextendible extension of $D(S)$ contains some chronology-violating region V'.

4. An Existence Theorem. With a definition in place, Earman et al. then conjecture that, under their formulation, there exist spacetimes that count as time machines. Here we prove this conjecture by showing that the well-

5. A result due to Krasnikov (2002) seems to indicate that, without the assumption of hole freeness, one can always find extensions of $D(S)$ bereft of closed timelike curves. For a discussion of whether hole freeness is a physically reasonable condition to place on spacetime, see Manchak 2009.

known example of Misner spacetime satisfies the conditions of the definition.[6] We have the following theorem.

Theorem 1. There exists an ESW time machine.

Proof. Let (M, g_{ab}) be Misner spacetime. So, $M = \mathbb{R} \times S$ and $g_{ab} = 2\nabla_{(a} t \nabla_{b)} \varphi + t \nabla_a \varphi \nabla_b \varphi$,, where the points (t, φ) are identified with the points $(t, \varphi + 2\pi n)$ for all integers n.

Let S be the spacelike slice $\{(t, \varphi) \in M : t = -1\}$. It can be easily verified that $D^+(S) = \{(t, \varphi) \in M : -1 \leq t < 0\}$. Let T be the compact set $\{(t, \varphi) \in M : t = -1/2\}$. So $T \subset D^+(S)$. Note that the set $\{(t, \varphi) \in M : t > 0\}$ is a chronology-violating region. Call it V. Clearly, $V \subset J^+[T]$. Thus, we have satisfied condition i of the definition of an ESW spacetime. For future reference, let N be the set $\{(t, \varphi) \in M : t \leq 0\}$.

Now let (M', g'_{ab}) be an inextendible extension of $D(S) = \{(t, \varphi) \in M : t < 0\}$ that does not contain closed timelike curves. We show that it must fail to be hole free. Now, for every $k \in [0, 2\pi]$, let γ_k be the null geodesic curve whose image is the set $\{(t, \varphi) \in M : \varphi = k \text{ and } -1 < t < 0\}$. Now, for each k, γ_k either has a future endpoint p_k or not. Clearly, for (M', g'_{ab}) to be inextendible, there is some k such that p_k exists. Let K be the set of all the endpoints p_k. We can extend the coordinate system used in Misner spacetime to a neighborhood $K' \subset M'$ of K. Under this coordinate system, we have $K = \{(t, \varphi) \in K' : t = 0\}$. For future reference, let the set N' be defined as $\{(t, \varphi) \in M' : t < 0 \text{ or } (t, \varphi) \in K\}$.

Next, we show that, for any distinct points $u, v \in K$, if $u \in J^-(v)$, then $v \notin J^-(u)$. It suffices to show that, for some $k \in [0, 2\pi]$, γ_k has no future endpoint (in that case, K cannot be a closed null curve). Assume that for all $k \in [0, 2\pi]$, there is a future endpoint p_k of γ_k in K. We show a contradiction. Consider any point $p_k \in K$ and a neighborhood $U_k \subset K'$ of p_k. Let $f_k : U_k \to \mathbb{R}$ be the function defined by $f_k(t, \varphi) = g'_{ab}(t, \varphi)(\partial/\partial\varphi)^a(\partial/\partial\varphi)^b$. Of course, when the domain of f_k is restricted to the set of points $(t, \varphi) \in U_k$ where $t \leq 0$, then $f_k(t, \varphi) = t$. The smoothness of g'_{ab} ensures that the boundary conditions $f_k(0, \varphi) = 0$ and $(\partial/\partial t) f_k(0, \varphi) = 1$ are satisfied. Clearly then, there must be an ε_k such that $f_k(t, \varphi) > 0$ for all $t \in (0, \varepsilon_k]$. Now let $\varepsilon : K \to \mathbb{R}$ be the function defined by $\varepsilon(p_k) = \varepsilon_k$. Note that the

6. For details concerning Misner spacetime, including a diagram, see Hawking and Ellis 1973, 171–174. Note, however, that because of the sign conventions used in that reference, the diagram there is an upside down representation of the version of Misner spacetime considered here.

smoothness of g'_{ab} allows us to choose our ε_k so that ε is a continuous function. Because K is compact, ε takes on a minimum value (call it $\varepsilon_{\min}$).[7] Next, let V' be the set $\{(t,\varphi) : 0 < t \leq \varepsilon_{\min}\}$. Clearly, on V', we have $g'_{ab}(\partial/\partial\varphi)^a(\partial/\partial\varphi)^b > 0$. Now let w be any point in V', and consider the curve $\gamma : I \to V'$ through w with tangent vector $\xi^a = (\partial/\partial\varphi)^a$ at every point. Because γ is contained entirely within V', we know that $g'\xi^a\xi^b > 0$. Thus, γ is a closed timelike curve, and we have a contradiction. So, we now know that, for any distinct points $u, v \in K$, if $u \in J^-(v)$, then $v \notin J^-(u)$.

Now let q be a point in K. Without any loss of generality, we may assume that $q \in K$ is the origin point $(0,0)$. Consider the spacelike surface Σ in (M', g'_{ab}), which is defined as the set $\{(t,\varphi) \in N' : -2\pi \leq t \leq 0 \text{ and } \varphi = -t\}$. Note that $q \in \Sigma$.

Now, we show that $D(\Sigma) \subseteq N'$. Let r be any point in $D(\Sigma)$. We show that r must also be in N'. It is easy to see that if $r \in D^-(\Sigma)$, then $r \in N'$. We turn to the other case: $r \in D^+(\Sigma)$. Assume $r \notin N'$. We show a contradiction. If $r \in D^+(\Sigma)$, then every past inextendible timelike curve through r must intersect Σ.[8] Since $r \notin N'$, every past inextendible timelike curve through r must intersect some $s \in K$. So we know that $s \in I^-(r)$ and $s \in D^+(\Sigma)$. Now let $\lambda : I \to K$ be the past inextendible null geodesic from s with tangent $(\partial/\partial\varphi)^a$.[9] Note that the image of γ is contained entirely within K. (It cannot enter the $t > 0$ region of K' for then γ must become spacelike. Similarly, γ cannot enter the $t < 0$ region of K' for then it must become timelike. So, it must remain in the $t = 0$ region, which by definition is just K.) On pain of contradiction, λ must intersect Σ. Since λ is contained within K, this means that $q \in J^-(s)$. Because $s \in I^-(r)$, this means that $q \in I^-(r)$.[10] Since $I^-(r)$ is open, we can find a point $q' \in K \cap J^-(q)$ in the neighborhood of q (distinct from q) such that $q' \in I^-(r)$. Clearly, we then can find a past-directed timelike curve from r to q' that fails to intersect q, and hence Σ (no past-directed timelike curve may enter and then leave the $t < 0$ region of M'). So, this means that $q' \in D^+(\Sigma)$. Now, let $\lambda' : I' \to K$ be the past inextendible null geodesic from q' with tangent $(\partial/\partial\varphi)^a$. On pain of contradiction, λ' must intersect Σ. Since λ' is contained entirely within K, this means that $q \in J^-(q')$. But we have shown above that, for any distinct points $u, v \in K$, if $u \in J^-(v)$, then $v \notin J^-(u)$. So, because q, q' are distinct

7. See Wald 1984, 425.

8. A past inextendible timelike or null curve has no past endpoint.

9. For details concerning geodesics, see Wald 1984, 41–47.

10. See Hawking and Ellis 1973, 183.

points in K and $q' \in J^-(q)$, we know that $q \notin J^-(q')$. However, this contradicts the fact that $q \in J^-(q')$. So, $D(\Sigma) \subseteq N'$.

Because $D(\Sigma) \subseteq N'$ and N' may be isometrically embedded, via the identity map, into Misner spacetime (M, g_{ab}), we know that there exists an isometric embedding $\theta : D(\Sigma) \to M$. It is easily verified that $D(\theta(\Sigma)) = N$. We have already shown that K cannot be a closed null curve. So clearly N' contains no closed null curves. Since $D(\Sigma) \subseteq N'$, there can be no closed null curves in $D(\Sigma)$ and hence none in $\theta(D(\Sigma))$. But there is a closed null curve in N. So $\theta(D(\Sigma)) \neq N$. So $D(\theta(\Sigma)) \neq \theta(D(\Sigma))$. This implies that (M', g'_{ab}) is not hole free, and we are done. QED.

5. Conclusion. So we have shown one sense in which there exist "time machines" within general relativity. We conclude with a few remarks about other ways one might interpret the result presented here.

Following Earman et al. 2009, we have assumed that spacetime is hole free and have then shown that certain initial conditions "force" the production of closed timelike curves. But instead we may have taken for granted that spacetime is free of closed timelike curves. In fact, this is routinely done (e.g., the singularity theorems of Hawking and Penrose [1970] proceed under this assumption). But then the logical structure of our result can be reworked to show that certain initial conditions "force" the production of "holes" in spacetime. So, in this way, the theorem demonstrates the existence of "hole machines" rather than "time machines."

We prefer to think of the theorem as a type of no-go result. It seems that some initial conditions force us to give up either (i) our intuition that spacetime is inextendible, (ii) our intuition that spacetime is hole free, or (iii) our intuition that spacetime is free of closed timelike curves.

REFERENCES

Earman, John (1995), *Bangs, Crunches, Whimpers, and Shrieks: Singularities and Acausalities in Relativistic Spacetimes*. New York: Oxford University Press.

Earman, John, Christopher Smeenk, and Christian Wüthrich (2009), "Do the Laws of Physics Forbid the Operation of Time Machines?", *Synthese* 169: 91–124.

Earman, John, and Christian Wüthrich (2004), "Time Machines", in E. N. Zalta (ed.), *Stanford Encyclopedia of Philosophy*, http://plato.stanford.edu/entries/time-machine/.

Geroch, Robert (1977), "Prediction in General Relativity", in John Earman, Clark Glymour, and John Stachel (eds.), *Foundations of Space-Time Theories*. Minnesota Studies in the Philosophy of Science, vol. 8. Minneapolis: University of Minnesota Press, 81–93.

Geroch, Robert, and Gary Horowitz (1979), "Global Structure of Spacetimes", in Stephen Hawking and Werner Israel (eds.), *General Relativity: An Einstein Centenary Survey*. Cambridge: Cambridge University Press, 212–293.

Hawking, Stephen, and George Ellis (1973), *The Large Scale Structure of Space-Time*. Cambridge: Cambridge University Press.

Hawking, Stephen, and Roger Penrose (1970), “The Singularities of Gravitational Collapse and Cosmology”, *Proceedings of the Royal Society of London A* 314: 529–548.

Krasnikov, S. V. (2002), “No Time Machines in Classical General Relativity”, *Classical and Quantum Gravity* 19: 4109–4129.

Manchak, John (2009), “Is Spacetime Hole-Free?”, *General Relativity and Gravitation* 41: 1639–1643.

Wald, Robert (1984), *General Relativity*. Chicago: University of Chicago Press.