Essay Review: *Topics in the Foundations of General Relativity and Newtonian Gravitation Theory*

John Byron Manchak*

David Malament, ***Topics in the Foundations of General Relativity and Newtonian Gravitation Theory***. Chicago: University of Chicago Press (2012), xii+349 pp., $55.00 (cloth).

The first portion of this book amounts to a formal introduction to differential geometry and the general theory of relativity (complete with problem sets and solutions). The second amounts to an investigation of special topics pursued by David Malament over the last 4 decades or so. In what follows, I intend to summarize the material. Along the way, I hope to offer some comments on the significance of the work.

Before I begin the survey of chapters, it might be helpful to say something about what the book is not. Although there is quite a bit of overlap between the first portion of the book and a standard graduate-level physics text, the two do not fully intersect (more below). In addition, the book does not consider many common topics in the philosophy of space-time physics. For example, there is no discussion of whether space and time are absolute or relational in character (cf. Sklar 1974; Friedman 1983). Finally, one does not find any precisely formulated conjectures of physical or philosophical interest that gesture toward future work (cf. Wald 1984; Earman 1995). It simply is not a book of that kind. Nonetheless, a number of topics developed by Malament in later chapters have generated a significant number of interesting results. (Indeed, I think the previous sentence is wildly understated.) Because Malament does not always do so, I hope to draw attention to some of these lines of inquiry below (a portion of which is still currently active).

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*To contact the author, please write to: Department of Philosophy, University of Washington, 361 Savery Hall, Box 353350, Seattle, WA 98195; e-mail manchak@uw.edu.

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Now, enough about what the book is not; I am ready to consider what the book *is*.

Chapter 1 provides an introduction to differential geometry. This presentation is self-contained; only some basic familiarity with point set topology and multivariable calculus is assumed. The material here follows a somewhat standard arrangement. One begins with the concept of a manifold and then considers various geometric structures defined on it: vector fields, tensor fields, derivative operators, metrics, and so on. Although the content of the chapter is not unusual, I do believe that the presentation here (and in what follows) is given with considerable care. An example might help to illustrate the point.

It is often useful to have at one's disposal a characterization of what it means to say that a smooth map $\Psi : S \rightarrow M$ is a (structure preserving) imbedding of one manifold S into another manifold M. When they are at all precise, the usual definitions include the condition that Ψ be injective. This requirement essentially prohibits the image of S from "intersecting itself." Another standard condition is that the inverse map $\Psi^{-1} : \Psi[S] \rightarrow S$ be continuous with respect to the relative topology on $\Psi[S]$. This requirement essentially prohibits the image of S from "almost intersecting itself." Malament provides a third condition that the linear map $(\Psi_p)_* : S_p \rightarrow M_{\Psi(p)}$ be injective for all points p in S. (This linear map is used to transfer arbitrary tensors at the point p to tensors at the point $\Psi(p)$.) This last requirement essentially prohibits the image of the tangent space at p from "intersecting itself."

One might wonder about the need for all three of the conditions. After all, each seems to rule out self-intersection of one kind or another, and, in addition, some texts do not even explicitly require the third. Is it possible that one or more of the conditions are redundant? No. Malament (94–95) methodically proceeds to show that any two of the three conditions are not sufficient to imply the other. (In each case, simple counterexamples are given to help the reader along.) I know of no other source that provides a comparable level of clarity on this particular matter. And the book is replete with this kind of attention to detail.

Chapter 2 presents the general theory of relativity. As mentioned above, many of the topics one finds in a graduate-level physics text are considered here. One begins with the notion of a (relativistic) space-time: a pair consisting of a manifold M and a metric g_{ab} of Lorentz signature $(1, -1, -1, -1)$ defined on M. One then moves on to the causal structure of space-time, the energy-momentum field, Einstein's equation, the initial value formulation, and so on. (There are 11 such subsections.) But there are still a number of standard topics that are not examined. In particular, the singularity theorems are not presented. Nor is the cosmic censorship hypothesis. And aside from the Friedmann models, there is no discussion of exact solutions to Einstein's equation. Readers interested in these subjects will want to consult Hawking and Ellis (1973), Wald (1984), and Joshi (1993).

But, because Malament does not undertake a systematic survey of the field, there is room in the chapter to present other (nonstandard) material. In fact, we get two glimpses of Malament's own contributions—one from the earliest stages of his career and another from the latest. That these contributions can be naturally presented in a chapter devoted solely to the most basic mathematical and physical structure of the theory is, in my opinion, significant. Let me say a bit more about each of them.

First, amid a discussion of the energy-momentum field T_{ab}, one finds an illuminating presentation of the various senses in which the geodesic principle—the statement that free massive point particles traverse timelike geodesics—does and does not follow from general relativity. The upshot (proposition 2.5.3) is taken from some of Malament's most recent work and shows that certain conditions on T_{ab}, which limit the distribution and flow of matter, turn out to be necessary for the principle to hold. The proposition has relevance to current debates regarding the status of the principle (Brown 2005; Weatherall 2011b; Tamir 2012).

Second, Malament presents (without proof) a highly nontrivial early result (proposition 2.2.4) that shows one sense in which the topological structure of space-time can be recovered from the causal structure. Because one is able to state (a portion of) the proposition without too much technical machinery, I will do so here. My hope is that, by formulating at least one theorem precisely, I will be able to give an accurate representation of Malament's somewhat unusual approach to philosophy. (Results in chaps. 3 an 4 will be stated more informally.)

Consider a (temporally oriented) space-time $(M,\ g_{ab})$. We define a two-place relation $\ll$ on the points in M: we write $p \ll q$ if and only if (iff) there exists a future-directed timelike curve $\gamma : [a,\ b] \rightarrow M$ such that $\gamma(a) = p$ and $\gamma(b) = q$. The idea is to have $p \ll q$ hold iff it is possible for a massive point particle to travel from p to q. Now, let $I^{-}(p)$ and $I^{+}(p)$ be the sets $\{q \in M : q \ll p\}$ and $\{q \in M : p \ll q\}$, respectively. We say that a space-time $(M,\ g_{ab})$ is *distinguishing* if, for all $p,\ q$ in M, the statements $I^{-}(p) = I^{-}(q)$ and $I^{+}(p) = I^{+}(q)$ each separately imply the statement $p = q$. Distinguishing space-times are causally "well behaved" in a somewhat weak sense; the condition essentially prohibits any timelike curves from "almost intersecting themselves."

We may now formulate the proposition: if $(M,\ g_{ab})$ and $(M',\ g'_{ab})$ are distinguishing space-times, then if $\varphi : M \rightarrow M'$ is a bijection such that, for all $p,\ q$ in M, $p \ll q$ iff $\varphi(p) \ll \varphi(q)$, then M and M' have the same topology. In other words, assuming the causal structure of space-time is sufficiently well behaved, then information concerning which points are causally related to which others is sufficient to recover the shape of the underlying manifold. In recent years, the proposition has served as the foundation for the causal set approach to quantum gravity (Dowker 2005; Sorkin 2005).

Stepping back a bit, one can appreciate some of the general features of a proposition of this kind. For one thing, it is proved entirely within the context of a well-confirmed scientific theory (in this case, general relativity). For another, the proposition amounts to a statement of some philosophical interest (in this case, one concerning reductionism, causation, etc.). The point I wish to emphasize is that, for these reasons, we have here a philosophical statement that is grounded firmly in some of our best science. Many of the results in chapters 3 and 4 (and in Malament's work more generally) are of the same character.

Chapter 3 considers a few special topics in general relativity and can be divided into two parts. In the first, we find a thorough presentation of the space-time of Gödel (1949)—a rather unusual exact solution of Einstein's equation. As Malament notes, the space-time is of interest because it helps us to understand the possibilities permitted by general relativity. Indeed, it seems that Gödel space-time is a counterexample to almost every reasonable statement concerning the general class of space-times (cf. Misner 1967).

Malament demonstrates a number of basic properties of Gödel space-time (M, g_{ab}). Here, I will limit myself to reviewing four. First, the space-time is in a state of constant, uniform rotation; at every point p in M, the rotation vector ω^a associated with the world lines of the major mass points is such that $\omega^a \neq 0$ and $\nabla_b \omega^a = 0$. This property will be of some use later on in the chapter. Second, Gödel space-time is globally homogeneous; for any two points p, q in M, there is an isometry (a metric preserving smooth map) $\psi : M \to M$ such that $\psi(p) = q$. Thus, any two points in the space-time are effectively indistinguishable from one another. Third, for any two points p, q in M we have $p \ll q$. So, any point is causally accessible to any other; in particular, any point is causally accessible to itself. Thus, there are closed (self-intersecting) timelike curves through every point allowing for "time travel" of a certain type. It is this property (along with Gödel's own considerations on the subject) that has received a great deal of attention in the philosophical literature (Stein 1970; Smeenk and Wüthrich 2011). Fourth, in Gödel space-time there are no closed timelike curves that are also geodesics; any "time travel" must be carried out along an accelerated curve. In fact, there is a lower bound on the total acceleration needed to complete such a journey (Malament 1985). Accordingly, one recent area of study concerns optimal time travel in the space-time (Manchak 2011; Natário 2012).

One final remark on this subsection: Malament exhibits the first and third properties mentioned above in an instructive diagram (fig. 3.1.1). It is notable because it improves on an influential representation of the same kind due to Hawking and Ellis (1973); in that reference, the vertical lines (which represent the world lines of the major mass points) are erroneously depicted as spacelike. A second diagram (fig. 3.1.2) also proves helpful in understanding the character of the timelike geodesics mentioned in the fourth property above.

In the second part of chapter 3, Malament considers the concept of rotation. In relativity theory, there is a clear (and absolute) sense in which a congruence of timelike curves with tangent field ξ^a can be said to be rotating at given point p; it is *rotating* at p iff $\omega^a \neq 0$ at p, where ω^a is the rotation vector field associated with ξ^a. But it is not at all clear what it means to say, for example, that a one-dimensional ring is rotating around a given (centered) axis.

Here, Malament sheds some light on the situation. He considers two possible criteria for nonrotation. One deals with the compass of inertia on the ring; gyroscopes might be employed to test for the presence of rotation. The other deals with the angular momentum of the ring; a light source might be employed to test for the Sagnac effect and thereby the presence of rotation. It turns out that these two criteria agree in some simple models such as Minkowski space-time. But Malament shows (proposition 3.2.4) that in Gödel space-time, the two criteria do not agree in general. It is possible that, according to one criterion, a given ring counts as rotating while, in another, it does not.

One can find an analogous (albeit simpler) situation concerning what it means to say that an ideal clock has "speeded up." I believe a few remarks by Geroch (1978, 133) on that particular situation can help place Malament's proposition in proper context:

> The Eskimos, so I understand, have over twenty-five different words for *snow*, words which distinguish the various subtle differences between various types of "snow." We have just one word. If light moved much more slowly in everyday terms then, I can guarantee, we would have something like twenty-five different words for *speeded up*. Similar remarks would apply to many other terms, including "speed of travel," "at rest," "simultaneous," "elapsed time," "spatial distance," "same position," "length," "straight," and so on. For these and many other everyday terms, one must take extreme care, in relativity, not to use them thoughtlessly.

Malament's work on the topic (described so far) amounts to the addition of the word "rotation" to the above list. To be sure, this is an interesting contribution. But one might wonder about the existence of some third (as yet unimagined) criterion that more adequately answers our classical intuitions. Is it possible that there is such a criterion? Under certain (seemingly weak) assumptions, no.

Malament considers three constraints one would like to see satisfied by any criterion under consideration. One constraint requires the criterion to agree "in the limit" with the clear notion of rotation at a point mentioned above. The second constraint requires the criterion to be nonvacuous (it cannot count all rings as rotating). Finally, the third constraint concerns the relative rotation of two arbitrary rings R_1 and R_2. It requires that if R_1 is counted as nonrotating and, in addition, R_2 is counted as nonrotating relative to R_1, then R_2 is also counted as nonrotating. Malament then shows (proposition 3.3.4) that no criterion exists that satisfies all three constraints in all space-times. It is a remarkable result and clarifies matters greatly.

Chapter 4 presents a version of Newtonian gravitation: Newton-Cartan theory (Cartan 1923, 1924; Friedrichs 1927). It is expressed in the formalism of differential geometry and therefore echoes the structure of general relativity rather closely. A (classical) space-time is a quadruple $(M, t_{ab}, h^{ab}, \nabla_a)$. Here, M is manifold, t_{ab} is a temporal metric of signature $(1, 0, 0, 0)$ on M, h^{ab} is spatial metric of signature $(0, 1, 1, 1)$ on M, ∇_a is a derivative operator on M, and the following orthogonality and compatibility conditions are satisfied: $h^{ab}t_{ab} = 0$, $\nabla_a t_{bc} = 0$, and $\nabla_a h^{bc} = 0$.

Newton-Cartan theory has been investigated by philosophers for some time now (Stein 1967; Glymour 1977; Earman 1989). Part of the interest concerns the fact that the theory has just enough such structure to allow one to speak meaningfully of absolute acceleration but not enough to do the same for absolute velocity. Other interest concerns the fact that the derivative operator ∇_a need not be "flat." (Formally, the Riemann curvature tensor $R^a{}_{bcd}$ associated with ∇_a need not vanish at every point in M.) So, there is a sense in which classical space-time is permitted to be curved just as it is in general relativity. For much of the chapter, Malament reviews various relationships between space-times with flat derivative operators and those with curved ones. Let me collect a few conditions together and say a bit more.

Consider a classical space-time $(M, t_{ab}, h^{ab}, \nabla_a)$ with a flat derivative operator. Let $\phi : M \to \mathbb{R}$ be the gravitational potential. Let $\rho : M \to \mathbb{R}$ be the mass-density field. We assume that Poisson's equation is satisfied: $\nabla_a \nabla^a \phi = 4\pi\rho$. Next, consider a point particle with mass m and tangent vector ξ^a. It has an acceleration vector $\xi^b \nabla_b \xi^a$. The gravitational force vector acting on the particle comes out as $-m\nabla^a \phi$. And if gravity is the only force present, the equation of motion $-\nabla^a \phi = \xi^b \nabla_b \xi^a$ must then be satisfied. Now one wonders whether it is possible to "geometrize" away the force of gravity. Trautman (1965) has shown that it is in the following sense (proposition 4.2.1): one can find a curved derivative operator $\overline{\nabla}_a$ on M, it too compatible with t_{ab} and h^{ab}, such that a timelike curve will satisfy the above equation of motion iff it is a geodesic according to $\overline{\nabla}_a$. Moreover, $\overline{\nabla}_a$ is unique. Malament interprets the state of affairs: "In the geometrized formulation of the theory, gravitation is no longer conceived of as a fundamental 'force' in the world but rather as a manifestation of spacetime curvature, just as in relativity theory. Rather than thinking of point particles as being deflected from their natural straight trajectories in flat spacetime, one thinks of them as traversing geodesics in curved spacetime" (269).

It turns out that one can also work in the other direction. There are two ways of doing so considered by Malament. Here, I will limit myself to Trautman's approach (proposition 4.2.5). Let $(M,\ t_{ab},\ h^{ab},\ \overline{\nabla}_a)$ be a classical spacetime with a curved derivative operator. To get off the ground, one needs the Riemann curvature tensor $R^a{}_{bcd}$ associated with $\overline{\nabla}_a$ to satisfy three conditions. Essentially, one ensures that Poisson's equation can eventually be satisfied while the other two serve as necessary integrability conditions. (More on these conditions below.) One can then find a flat derivative operator ∇_a on M, it too compatible with t_{ab} and h^{ab}, and a gravitational potential $\phi : M \rightarrow \mathbb{R}$ such that a timelike curve will satisfy the equation of motion $-\nabla^a \phi = \xi^b \nabla_b \xi^a$ iff it is a geodesic according to $\overline{\nabla}_a$. But, unlike the geometrization case, the flat derivative operator ∇_a and gravitational potential ϕ that one finds will not be a unique pair. It is this fact that captures a clear sense in which the Newtonian gravitational potential is a "gauge" quantity.

Malament devotes a significant portion of the chapter to the three conditions on $R^a{}_{bcd}$ mentioned above. In one subsection, they are interpreted geometrically. Elsewhere, there is an investigation as to when one might expect the conditions to be satisfied. Roughly, one of the integrability conditions (4.2.20) will be satisfied if $(M,\ t_{ab},\ h^{ab},\ \overline{\nabla}_a)$ is homogeneous and isotropic or if it is asymptotically flat. The two other conditions (4.2.18, 4.2.19) will be satisfied if $(M,\ t_{ab},\ h^{ab},\ \overline{\nabla}_a)$ is, in a precise sense, the result of a limiting procedure applied to a one-parameter family of general relativistic spacetimes. The procedure essentially lets the speed of light go to infinity (Künzle 1976; Ehlers 1981; Malament 1986).

In the later portion of the chapter, Malament considers an old "paradox" concerning Newtonian cosmology. It amounts to this: if one were to use a standard (high school level) formulation of Newtonian mechanics, one would seem to be led to logical inconsistencies concerning the gravitational field in a homogeneous universe (Norton 1993). But Malament shows (proposition 4.4.3) that the paradox is nothing more than a relic of the formulation and dissolves completely in the Newton-Cartan theory.

Let me conclude my remarks on this chapter by saying a word or two about the literature associated with the Newton-Cartan theory. Unlike general relativity, there simply has not been a contemporary, systematic treatment of the subject available anywhere. Malament has done a great service to the community by providing one. The exposition is lucid and thorough. Earlier versions of this presentation have already been used to clarify a number of topics in general philosophy of science such as empirical indistinguishability (Bain 2004) and explanation (Weatherall 2011a). I suspect that work done in this chapter will open the way for even more researchers to consider the rich set of ideas examined here.

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