# Dedekind's Contributions to the Foundations of Mathematics

Erich H. Reck, November 2007 — Draft; please do not quote!

Richard Dedekind (1831-1916)

It is widely acknowledged that Dedekind was one of the greatest mathematicians of the nineteenth-century, as well as one of the most important contributions to number theory and algebra of all time. Any comprehensive history of mathematics will mention him for, among others: his invention of the theory of ideals and his investigation of the notions of algebraic number, field, module, lattice, etc. (see, e.g., Boyer & Merzbach 1991, Kolmogorov & Yushkevich 2001, Alten et al. 2003). Dedekind's more foundational work in mathematics is also widely known, at least in parts. Often acknowledged in that connection are: his analysis of the notion of continuity, his construction of the real numbers in terms of Dedekind-cuts, his formulation of the Dedekind-Peano axioms for the natural numbers, his proof of the categoricity of these axioms, and his contributions to the early development of set theory (Ferreirós 1999, Jahnke 2003).

While many of Dedekind's contributions to mathematics and its foundations are thus common knowledge, they are seldom discussed together. In particular, his mathematical writings are often treated separately from his foundational ones. This entry provides a comprehensive survey of his contributions, in all of his major writings, with the goal of making their close relationships apparent. It will be argued that foundational concerns are at play throughout Dedekind's work, so that any attempt to distinguish sharply

between his "mathematical" and "foundational" writings is artificial and misleading. Another goal of the entry is to establish the continuing relevance of his contributions to the philosophy of mathematics. Indeed, their full significance has only started to be recognized, as should become evident along the way. This is especially so with respect to methodological and epistemological aspects of Dedekind's work, which shape and ground the logical and metaphysical views that emerge in his writings.

The entry is divided into the following sections:

1. Biographical Information
2. The Foundations of Analysis
3. The Foundations of Arithmetic
4. The Rise of Modern Set Theory
5. Logicism and Structuralism
6. Algebraic Number Theory
7. Other Mathematical Work
8. Methodology and Epistemology
9. Concluding Remarks

At the end of each section, a selection of relevant secondary literature is listed, both to direct the reader to further resources and to emphasize my large debt to that literature.

## 1. Biographical Information

Richard Dedekind was born in Brunswick (Braunschweig), a city in northern Germany, in 1831. Much of his education took place in Brunswick as well, where he first attended school and then, for two years, the local technical university. In 1850, he transferred to the University of Göttingen, a center for scientific research in Europe at the time. Carl Friedrich Gauss, generally considered one of the greatest mathematicians of all time, taught in Göttingen, and Dedekind became his last graduate student. He wrote a dissertation in mathematics under Gauss, finished in 1852. As was customary, he also wrote a second dissertation (Habilitation), completed in 1854, shortly after that of his friend and colleague Bernhard Riemann. Dedekind stayed in Göttingen for four more

years, as an unsalaried lecturer (Privatdozent).  During that time he was strongly influenced by P.G. Lejeune-Dirichlet, another renowned mathematician in Göttingen, and by Riemann, then a rising star.  (For each of them, Dedekind later did important editorial work, also posthumously.)  In 1858, he moved to the Polytechnicum Zürich, Switzerland, to take up his first salaried position.  He returned to Brunswick in 1862, where he became professor at the local university and taught until his retirement in 1896.  In this later period, he published most of his major works; he also had further interactions with important mathematicians.  Among others, he was in correspondence with George Cantor, collaborated with Heinrich Weber, and developed an intellectual rivalry with Leopold Kronecker.  He stayed in his hometown until the end of his life, in 1916.

Dedekind's main foundational writings are:  "Stetigkeit und Irrationale Zahlen" (1872) and "Was sind und was sollen die Zahlen?" (1888).  Equally important, according to historians of mathematics, is his work in algebraic number theory.  That work was first presented in an unusual manner: as supplements to Dirichlet's "Vorlesungen über Zahlentheorie".  The latter was edited by Dedekind in the form of lecture notes, initially under Dirichlet's supervision, and published in a number of editions.  It is in his supplements to the second edition, from 1871, that Dedekind's famous theory of ideals is introduced; he then modified and expanded it several times, until the fourth edition of 1893 (Lejeune-Dirichlet 1893, Dedekind 1964).  A relatively early version of Dedekind's theory was also published separately, in a French translation (Dedekind 1877).  Further works by Dedekind include: a long article on the theory of algebraic functions, written jointly with Heinrich Weber (Dedekind 1882); a variety of shorter pieces in algebra, number theory, complex analysis, probability theory, etc.  All of these were re-published, with selections from his Nachlass, in Dedekind (1930-32).  Finally, lecture notes from some of his own classes were made available later (Dedekind 1981, 1985), as were additional selections from his Nachlass (Dedekind 1982).

As this brief chronology of Dedekind's life and publications indicates, he was a wide-ranging and very creative mathematician, although he tended to publish slowly and carefully.  It also shows that he was part of a distinguished tradition in mathematics, extending from Gauss and Dirichlet through Riemann, Dedekind himself, Cantor, and

Weber in the nineteenth century, on to David Hilbert, Emmy Noether, B.L. van der Waerden, Nicolas Bourbaki, and others in the twentieth century. With the partial exceptions of Riemann, Cantor, and Hilbert, these mathematicians did not publish explicitly philosophical treatises. On the other hand, all of them were very sensitive to foundational issues understood in a broad sense, including the choice of basic concepts, the kinds of reasoning to be used, and the presuppositions build into them. As a consequence, one can find philosophically pregnant remarks sprinkled through their mathematical works, as exemplified by the prefaces to Dedekind (1872, 1888).

Not much is known about other intellectual influences on Dedekind. He did not align himself explicitly with a particular philosopher or philosophical school, in addition to the mathematical tradition mentioned. Indeed, little is known about which philosophical texts Dedekind read or which corresponding classes he might have attended as a student, if any. A rare piece of information we have in this connection is that he became aware of Gottlob Frege's most philosophical work, *Die Grundlagen der Arithmetik*, only after having settled on his own basic ideas (Dedekind 1888a, preface to the $2^{nd}$ edition). In Dedekind's short biography of Riemann (Dedekind 1876a) he also displays familiarity with the post-Kantian philosopher and educator J.F. Herbart, professor in Göttingen from 1833 to 1841, who is mentioned as an influence on Riemann. Then again, German intellectual life at the time was saturated with Kantian views, including claims about the essential role of intuition for mathematics, and with various responses to them. It is to be expected that Dedekind was familiar with at least some related discussions.

LITERATURE: No book-length biography of Dedekind exists so far. For shorter accounts, see Jourdain (1916), Biermann (1971), Dugac (1976), Scharlau (1981), Mehrtens (1982), Ewald (1996), Laugwitz (1996), Stein (1998), Ferreirós (1999), Sieg & Schlimm (2005).


## 2. The Foundations of Analysis

The issues addressed in Dedekind's "Stetigkeit und Irrationale Zahlen" ("Continuity and Irrational Numbers") grow out of the "rigorization" and "arithmetization" of analysis (the mathematical theory) in the early nineteenth century. The roots of these issues go deeper,

however—all the way down, or back, to the discovery of incommensurable magnitudes in Ancient Greek geometry (Jahnke 2003, chapter 1). The Greeks' response to this startling discovery culminated in Eudoxos' famous theory of proportionality and of ratios, presented in Chapter VII of Euclid's *Elements*. This theory brought with it a sharp distinction between discrete numbers and continuous magnitudes or quantities, thus leading to the traditional view of mathematics as a whole as the science of number, on the one hand, and of magnitude, on the other. Dedekind's first foundational work concerns, at bottom, the relationship between the two sides of this dichotomy.

An important part of the dichotomy, as traditionally understood, was that magnitudes, and then also ratios of them, were not systematically thought of as numerical entities (with arithmetic rules defined on them), but in a more concrete geometric way (as lengths, areas, angles, etc. and proportions between them). More particularly, while Eudoxos' theory provides a contextual criterion for the equality of ratios, it does not provide for a definition of the ratios themselves, so that they are not conceived of as independent entities (Stein 1990). Such features do little harm with respect to empirical applications of the theory. They lead to inner-mathematical tensions, however, for instance when considering solutions to certain kinds of algebraic equations (some of which could be represented numerically but others only geometrically). This tension came increasingly to the fore in the mathematics of the early modern period, especially after Descartes' integration of algebra and geometry. What was called for, in response, was a unified treatment of discrete numbers and continuous quantities.

More directly, Dedekind's essay was tied to the arithmetization of analysis in the first half of the nineteenth century—pursued by Cauchy, Bolzano, Weierstrass, and others—which in turn was a reaction to tensions within the Differential and Integral Calculus, introduced earlier by Newton, Leibniz, and their followers (Jahnke 2003, chs. 3-6). As is well known, the inventors of the Calculus relied on appeals to "infinitesimal" quantities, typically backed up by geometric or even mechanical considerations. This had come to be seen as problematic, however. The early "arithmetizers" found a way to avoid infinitesimals (in terms of the epsilon-delta characterization of limit processes familiar from current introductions to the Calculus). But this again, or even more, led to the need

for an acceptable and fully explicit characterization of irrational quantities conceived of as numerical entities, thus also for a unified treatment of rational and irrational numbers.

Dedekind faced this need directly, also from a pedagogical perspective, when he started to teach classes on the Calculus, in 1858, at Zürich (Dedekind 1972, preface). Moreover, the goal for him, from early on, was not just to supply an acceptable, explicit, and unified treatment of both rational and irrational numbers—he also wanted to do so in a way that established the independence of analysis from mechanics and geometry, indeed from all intuitive notions (ibid.). This reveals a further philosophical motivation for Dedekind's work on the foundations of analysis, although one not unconnected with the mathematical background mentioned. Also, it is not hard to see an implicit anti-Kantian thrust in his approach. Finally, the way in which to achieve all of these objectives was to relate analysis and arithmetic closely to each other, indeed to reduce the former to the latter.

While the basic idea of tying analysis closely to arithmetic, as opposed to geometry, was not new a the time—Dedekind shared it with, and probably adopted it from, his teachers Gauss and Dirichlet (Ferreiros 1999, ch. 4)—the particular manner in which he then proceeded was quite original. The crucial issue, or the linchpin, for him was the notion of continuity. To get clearer about that notion, Dedekind compared the system of rational numbers with the system of points on a geometric line. Once a point of origin, a unit length, and a direction have been picked for the latter, the two systems are directly relatable: each rational number can be mapped, in a unique and order-preserving way, onto a point on the line. A further question then arises: Does each point on the line correspond to a rational number? Crucially, that question can be reformulated in terms of Dedekind's idea of "cuts" and, thus, as involving only the rational numbers, so that any intuitive understanding of continuity can be put aside. Namely, if we divide the whole system of rational numbers into two disjoint parts, preserving their order, is each such division determined by a rational number? The answer is No, since some correspond to irrational numbers (e.g., the cut consisting of $\{x : x^2 < 2\}$ and $\{x : x^2 \geq 2\}$ corresponds to $\sqrt{2}$). In this sense, the system of rational numbers is not line-complete, or not continuous.

For our purposes, several aspects of Dedekind's procedure, initially and in the next few

steps, are important (compare Ferreirós 1999, ch. IV, also Cook 2005). As mentioned, Dedekind starts by considering the system of rational numbers as a whole. Noteworthy here are two aspects: Not only does he accept this whole as a "actual infinity", in the sense of a complete infinite set that is treated as a mathematical object in itself; he also considers it "structurally", as an example of a linearly ordered system on which the operations of addition and multiplication are defined. In his next step—and proceeding further along set-theoretic and structuralist lines—Dedekind introduces the set of arbitrary cuts on his initial system, thus working, essentially, with the bigger and more complex infinity of all subsets of the rational numbers (the full power set); and he then endows the set of all cuts, in turn, with a linear ordering and with operations of addition and multiplication, thus constructing a totally new "number system".

Next, two properties of the system of cuts are established: The rational numbers can be embedded into it, in a way that respects the ordering and the arithmetic operations defined on those numbers; in modern terminology, a corresponding homomorphism exists. And the system of cuts is continuous, in the sense defined above, with respect to its own order. There is one more step in Dedekind's construction procedure. Namely, for each cut—those corresponding to rational numbers, but also those corresponding to irrational magnitudes—he "creates" a new object, a "real number" determined by the cut (but distinct from it). What that step does, among others, is to provide the long missing unified criterion of identity for rational and irrational numbers, both treated as elements in an encompassing number system (isomorphic to, but distinct from the system of cuts). Finally, Dedekind shows that, along such lines, explicit and straightforward proofs of various facts about the real numbers can be given, including ones that had been accepted without satisfactory proof so far. These include: basic rules of operation with square roots; the fact that any increasing bounded sequence of real numbers has a limit value (a result equivalent to the more well-known mean value theorem).

Dedekind's published this construction and theory of the real numbers only in 1872, fourteen years after developing the basic ideas on which it relies. It was not the only construction proposed at the time. Various mathematicians addressed this issue, including: Weierstrass, Hankel, Méray, Heine, Cantor, Thomae, and somewhat later,

Frege (Jahnke 1999, ch. 10, Boyer & Merzbach 1991, ch. 25). Most familiar among those alternative treatments is probably Cantor's, also published in 1872. Instead of using "Dedekind-cuts", Cantor works with (equivalence classes of) Cauchy sequences or rational numbers. The system of such (classes of) sequences can also be shown to have the desired properties, including continuity. Cantor, like Dedekind, starts with the infinite set of rational numbers; and Cantor's construction again relies essentially on the full power set of the rational numbers, here in the form of arbitrary Cauchy sequences. In such set-theoretic respects the two treatments are thus equivalent. What sets apart Dedekind's treatment of the real numbers, from Cantor's and all the others, is the special attention given by him to the core notion of continuity. His treatment is also more maturely, and elegantly, structuralist, in a sense to be spelled out further below.

LITERATURE: A more detailed summary of Dedekind (1872) can be found in Cook (2005). For further background and discussion, compare Dugac (1976), Gardies (1984), Stein (1990), Boyer & Merzbach (1991), Ferreirós (1999), and Jahnke (2003).


## 3. The Foundations of Arithmetic

Providing an explicit, precise, and fruitful definition of the real numbers constitutes a major step towards completing the arithmetization of analysis. Further reflection on Dedekind's procedure (and similar ones) leads to a new question, however: What exactly is involved in it if thought through fully—what does his definition and treatment of the real numbers rely on, ultimately? As noted, Dedekind starts with the system of rational numbers; and he uses a set-theoretic procedure to construct the new system of cuts out of them. This leads to two sub-questions: First, how exactly are we to think about the rational numbers in this connection? Second, can anything further be said about the set-theoretic procedures and the assumptions behind their employment?

In his published writings, Dedekind does not provide a direct and full answer to our first sub-question. What suggests itself, from a contemporary point of view, is that he relied on the following assumption: The rational numbers can be conceived of in terms of the natural numbers, together with some set-theoretic ideas. And in fact, in Dedekind's

Nachlass explicit sketches of two now familiar constructions can be found: that of the positive and negative integers as (equivalence classes of) pairs of natural numbers; and that of the rational numbers as (equivalence classes of) pairs of integers (see Schlimm 2000, Sieg & Schlimm 2005). It seems that these constructions were familiar enough at the time, in one form or another, for Dedekind not to feel the need to publish his sketches. (There is also a direct parallel to the construction of the complex numbers as pairs of real numbers in Gauss' works, and to the use of equivalence classes when developing modular arithmetic already in Dedekind 1857. For the latter, compare Dugac 1976; for more on the early development of Dedekind's views, see again Sieg & Schlimm 2005.)

This leads to the following conclusion: All that is needed for analysis, including the real numbers, can be constructed out of the natural numbers by set-theoretic means. But now the question becomes: Do we have to take the natural numbers themselves as simply given; or can anything further be said about those numbers, perhaps by reducing them to something even more fundamental? Many mathematicians in the nineteenth century were willing to assume the former. A well-known example is Leopold Kronecker, for whom the natural numbers are "given by God", while the rest of arithmetic and analysis is "made by mankind" (Ferreirós 1999, ch. 4). In contrast, Dedekind, and independently Frege, pursued the latter option—they attempted to reduce the natural numbers and arithmetic to "logic". This is the main goal of "Was sind und was sollen die Zahlen?" ("The Nature and Meaning of Numbers", or more directly, "The Nature and Purpose of Numbers"). Another goal is to answer the question left open above: whether more can be said about the set-theoretic tools used. For Dedekind, again like for Frege, these tools are part of "logic" as well. But then, what are the basic notions of logic?

Dedekind's answer to this last question is: basic for logic are the notions of object ("Ding"), set (or system, "System"), and function (mapping, "Abbildung"). These notions are, indeed, fundamental for human thought—they are applicable in all domains, indispensable in exact reasoning, and cannot be reduced further (Ferreirós 1999, ch. 4). While irreducible, thus not definable in terms of anything more basic, the basic logical notions are nevertheless capable of being elucidated, thus of being understood better. Part of their elucidation consists in observing what can be done with them, including how

arithmetic can be developed in terms of them. (More on other parts in the next section.) For Dedekind, the latter starts with the consideration of infinite sets, similarly to the case of the real numbers but in a generalized, more systematic way.

Dedekind does not just assume, or simply postulate, the existence of infinite sets—he tries to prove it, namely by considering "the totality of all things that can be objects of my thought" and by arguing that this "set" is infinite (Dedekind 1888a, section v). He also does not just presuppose the concept of infinity—he defines it (utilizing his three basic logical notions, as well as the related, definable notions of subset, union, intersection, etc.) as follows: A set of objects is infinite—now often, "Dedekind-infinite"—if it can be mapped one-to-one onto a proper subset of itself. Moving a step closer to arithmetic, we then get the notion of a "simple infinity". Introducing that notion involves Dedekind's original idea of a "chain". In contemporary terminology, a chain is the minimal closure of a set $A$ in a set $B$ containing $A$ under a function $f$ on $B$ (where being "minimal" is understood via the general notion of intersection).

What it means to be simply infinite can now be captured in four conditions: Consider a set $S$, a function $f$ on $S$, a subset $N$ of $S$ (possibly equal to $S$), and an element $1$ of $N$; then $N$ (or really the system consisting of $N$, $f$, and $1$) is called simply infinite if (i) $f$ maps $N$ into itself, (ii) $N$ is the chain of $\{1\}$ in $S$ under $f$, (iii) $1$ is not in the image of $N$ under $f$, and (iv) $f$ is one-to-one. While at first perhaps unfamiliar, it is not hard to show that these four conditions are a notational variant of the Peano Axioms. In particular, condition (ii) is a version of the axiom of mathematical induction. These axioms are thus properly called the Dedekind-Peano Axioms. (Peano, who published his corresponding work in 1889, acknowledged Dedekind's priority; see Ferreirós 1999, 2005.) As is also readily apparent, any simple infinity will consist of a first element $1$, a second element $f(1)$, a third $f(f(1))$, then $f(f(f(1)))$, etc., just like any model of the Dedekind-Peano Axioms.

Given all of these preparations, the construction of the natural numbers can now proceed as follows: First, Dedekind proves that every infinite system contains a simply infinite subsystem. Then he shows, again in contemporary terminology, that any two simply infinite systems, or any two models of the Dedekind-Peano Axioms, are isomorphic, so

that the axiom system is categorical. Third, he notes that, because of this fact, exactly the same truths hold for all simple infinities; or closer to Dedekind's actual way of putting this point, any truth about one of them can be translated, via the isomorphism, into a corresponding truth about the other. In that sense, all of them are equivalent—from a logical point of view, each is as good as any other. (In other words again, all models of the Dedekind-Peano Axioms are "logically equivalent", which means that the axiom system is "semantically complete"; see Awodey & Reck 2002 and Reck 2003a).

As a fourth step, Dedekind again appeals to the notion of "creation". Starting with some particular simple infinity constructed initially—it doesn't really matter which one we start with, given their isomorphism—he "creates" a new object corresponding to each of its elements, thus introducing a special simple infinity, "the natural numbers". As we already saw, this last move has a parallel in the case of the real numbers. But in the present case Dedekind is more explicit about the following: The identity of the newly created objects is determined completely by what is true of them arithmetically, i.e., by the truths that are transferable, or invariant, in the sense explained above. Put differently, the natural numbers, as constructed by Dedekind, are characterized by their "relational" or "structural" properties alone—unlike the elements in other simply infinite systems, they have no non-arithmetic, or "foreign", properties (Reck 2003a).

What Dedekind has captured, along such structuralist lines, is the natural numbers conceived of as finite "ordinal" numbers (or counting numbers: the first, the second, etc.). He immediately adds an explanation of how the usual use of the natural numbers as finite "cardinal" numbers (answering to the question: how many?) can be recovered. This is done by considering initial segments of the number series and using them as tallies, i.e., by asking for any set which such segment can be mapped one-to-one onto it, if any, thus measuring its "cardinality". (A set turns out to be finite if and only if there exists such a segment.) Dedekind concludes his essay by showing how several basic, and formerly unproven, arithmetic facts can be proved along such lines. Especially significant are his purely "logical" justifications of the methods of proof by mathematical induction and of definition by recursion (based on his theory of chains).

LITERATURE: For a comparison of Dedekind (1888) with Peano's work, see Ferreirós (2005); compare Dugac (1976), Gillies (1982), Belna (1996), Tait (1997), Ferreirós (1999), Potter (2000), Awodey & Reck (2002), Reck (2003a), Sieg & Schlimm (2005).

## 4. The Rise of Modern Set Theory

As we saw earlier, set-theoretic assumptions and techniques are already used in Dedekind's "Stetigkeit und Irrationale Zahlen". Thus, the system of rational numbers is assumed to be an infinite set. The set of arbitrary cuts of rational numbers, constructed out of it, is then also treated as an infinite set. When supplied with an order and arithmetic operations on its elements, the latter gives rise to a new number system. Parallel moves can be found in the sketches, from Dedekind's Nachlass, for how to construct the integers and the rational numbers. Here again, we start with infinite sets; and new number systems are constructed out of them set-theoretically (although in those cases the full power set of an infinite set is not needed). Finally, Dedekind employs the same kinds of techniques in his other mathematical work (his development of modular arithmetic, his definition of ideals as infinite sets, etc.), as we will see more below. It should be emphasized that the application of such techniques was quite novel, and bold, at the time. While a few mathematicians, such as Cantor, used related ones, many others, like Kronecker, rejected them. In fact, by working seriously with actual infinities Dedekind took a stance incompatible with that of his teacher Gauss, who had allowed for the infinite only as a "manner of speaking" (Edwards 1983, Ferreirós 1999, ch. 7).

What happens in Dedekind's "Was sind und was sollen die Zahlen?", in connection with his logicist construction of the natural numbers, is that this employment of set-theoretic techniques is lifted to a new level of explicitness and clarity. Thus, Dedekind not only presents set-theoretic definitions of further mathematical notions—he also adds a systematic reflection on the means used thereby (and expands that use in certain respects, as discussed more below). As such, this essay constitutes an important step in the rise of modern set theory. As we saw, Dedekind presents the notion of set, together with the notions of object and function, as fundamental for human thought. Here an object is meant to be anything for which it is determinate how to reason about it, including having

definite criteria of identity (Tait 1997, Reck 2003a). Sets are particular objects, ones about which we reason by considering their elements. The latter is, in fact, all that matters about sets. In other words, sets are to be identified extensionally, as Dedekind is one of the first to emphasize. (Even as important a contributor to set theory as Bertrand Russell will struggle with this point well into the twentieth century; see Reck 2003b.)

Functions are also to be conceived of extensionally, as ways of correlating the elements of sets. But unlike in axiomatic set theory, Dedekind does not reduce functions to sets. (Not unreasonably, the ability to map one thing onto another, or to represent one by the other, is taken as fundamental for human thought; Dedekind 1888a, Preface). Another aspect of Dedekind's views about functions is that, with respect to the range of functions considered, he allows for arbitrary injective correlations between sets of numbers, indeed between sets of objects in general. He thus rejects restrictions of the notion of function to cases presented by formulas, representable in intuition (via their graphs), or decidable by a formal procedure. In other words, Dedekind works with a general notion of function, parallel to his general notion of set. In this respect he adopts, and expands further, the view of another of his teachers: Dirichlet (Ferreirós 1999, ch. 7, Sieg & Schlimm 2005).

Such general notions of set and function, combined with the acceptance of the actual infinite that gives them bite, were soon attached by finitistically and constructively oriented mathematicians such as Kronecker. Dedekind defended this feature of his approach explicitly, by pointing, in particular, towards its fruitfulness (Dedekind 1888a, first footnote, also Edwards 1983, Ferreirós 1999, ch. 7). However, he came to see another feature as problematic—his use of a "naïve" notion of set. According to that notion, any collection of objects counts as a set (i.e., an unrestricted comprehension principle is assumed). We already encountered a specific way in which this comes up in Dedekind's work: "Was sind und was sollen die Zahlen?" relies on introducing "the totality of all things that can be objects of my thought" (his "universal set"); and then arbitrary subsets of that totality are considered (thus a general "Aussonderungsaxiom" is used). But then, any collection of objects we can think of counts as a set.

A general point to observe in this connection is that Dedekind has gone beyond

considering sets of numbers (and sets of sets of numbers, etc.), as he did earlier; sets of other objects are now within his purview as well (similarly for functions).  This is a significant extension of the notion of set, or of its application; but it is not problematic in itself.  A more problematic point is the way in which Dedekind's all-encompassing totality is introduced, namely by reference to "human thought".  This aspect leads to the worry of whether psychologistic features are involved (more on that worry later).  The most problematic aspect—and the one Dedekind took rather seriously himself—is a third one: his set theory is subject to Russell's Antinomy.  (If any collection of objects counts as a set, then also Russell's collection of all sets that do not contain themselves; this leads quickly to a contradiction.)  Dedekind seems to have found out about this antinomy in the 1890s, from Cantor who had discovered it independently.  The discovery shocked him.  Not only did he delay republication of "Was sind und was sollen die Zahlen?" because of it; pending a resolution, he even expressed doubts about "whether human thinking is fully rational" (Dedekind 1930-32, Vol. 3, p. 449, also Reck 2003a, fn. 9).

Russell's antinomy establishes that Dedekind's original conception of sets is untenable.  However, this does not invalidate his other contributions to set theory.  Dedekind's analysis of continuity, the use of Dedekind cuts in the construction of the real numbers, the definition of being Dedekind-infinite, the formulation of the Dedekind-Peano Axioms, the proof of their categoricity, the analysis of the natural numbers as finite ordinal numbers, the corresponding justification of mathematical induction and recursion, and more basically, the insistence on an extensional and general notion of set and function, as well as the acceptance of the actual infinite—all of these contributions can be isolated from Russell's antinomy.  As such, they are all built centrally into contemporary axiomatic set theory, model theory, recursion theory, and related parts of logic.

And there were further contributions to set theory by Dedekind.  These do not occur in his published writings, but in his correspondence.  Especially significant is his correspondence with Cantor (Noether & Cavaillès 1937, Meschkowski & Nilson 1991), which started, in 1872, after the two mathematicians had met in person.  It contains a discussion of Cantor's and Dedekind's respective constructions of the real numbers.  But more than that, their letters amounts to a joint exploration of the notions of set and

infinity (Ferreirós 1993, also Ferreirós 1999, ch. 7). Among the contributions Dedekind makes in this connection are the following: He impressed Cantor with a proof that not only the set of rational numbers, but also that of all algebraic numbers is countable. This led, at least in part, to Cantor's further study of infinite cardinalities and his discovery, soon thereafter, that the set of all real numbers is not countable. And Dedekind provided a proof of the Cantor-Bernstein Theorem (that between any two sets which can be injected into each other one-to-one there exists a bijection, so that they have the same cardinality), another basic part of the modern theory of transfinite cardinals.

Finally, in the further development of set theory early in the twentieth century it became clear that some of Dedekind's original procedures and results could be generalized in important ways. Perhaps most significantly, Zermelo succeeded in extending Dedekind's analysis of mathematical induction and recursion (in terms of the theory of chains) to the higher infinite, thus developing, and establishing more firmly, Cantor's theory of transfinite ordinals and cardinals. Looking back at all of these contributions, it is no wonder that Zermelo—who knew the relevant history well—considered the modern theory of sets as having been "created by Cantor and Dedekind" (quoted in Ferreirós 2007; see also Ferreirós 1999, ch. 7).

LITERATURE: For a rich study of the rise of modern set theory, including Dedekind's role in it, see Ferreirós (1999); for a brief overview, compare Ferreirós (2007); see also Dugac (1976), Ferreirós (1993), Tait (1997), Reck (2003a), and Sieg & Schlimm (2005).

## 5. Logicism and Structuralism

So far Dedekind's contributions in his overtly foundational works were in focus. We reviewed his set-theoretic reconstructions of the natural and real number, as basic mathematical entities. We also sketched his role in the rise of modern set theory. Along the way, philosophical issues came up; but a more extended reflection on them seems called for, especially a reflection on Dedekind's "logicism" and "structuralism".

Like for Frege, the other main logicist in the nineteenth century, for Dedekind "logic" is more encompassing than often assumed today (as comprising only first-order logic). For

both thinkers, the notions of object, set, and function are absolutely basic for human thought and, as such, fall within logic's range. Consequently, each of them developed a version of set theory (a theory of "systems", "classes", or "extensions"), as part of logic. Also for both, logic, even in this encompassing sense, is independent of intuitive considerations, and specifically of geometry (understood itself as grounded in intuition). What reducing arithmetic and analysis to logic was meant to establish, then, is that those fields too are independent of intuition (Reck 2003a, Demopoulos & Clark 2005).

The view that arithmetic and analysis are independent of geometry, since they fall within the realm of purely logical thought, was not entirely new at the time—Gauss and Dirichlet already held such a view, as mentioned in the case of analysis. But what Dedekind and Frege added were particular, detailed reductions of analysis to arithmetic, complemented, or grounded, by parallel reductions of arithmetic to logic, on the one hand, and by systematic elaborations of logic, on the other. Moreover, as Dedekind's work was better known than Frege's at the time, perhaps because of his greater reputation as a mathematician, he was seen as the main representative of "logicism" by interested contemporaries, such as C. S. Peirce and Ernst Schröder (Ferreirós 1999, ch. 7).

Besides such commonalities between Dedekind's and Frege's versions of logicism, they also agreed on a more general methodological principle, encapsulated in the following remark by Dedekind: In science, and especially in mathematics, "nothing capable of proof ought to be accepted without proof" (Dedekind 1888a, Preface). For both, this principle out to be adhered to not so much because it increases certainty in mathematical results. Rather, it is often only by providing an explicit, careful proof for a result that the assumptions on which it depends become evident, thus also its range of applicability. Dedekind and Frege had learned this lesson from the history of mathematics, especially the recent history of geometry, algebra, and the Calculus (compare Tappenden 2006).

Beyond where they agree, it is instructive to consider also some of the differences between Dedekind and Frege. First, and put in modern terminology, a major difference is that Frege tends to focus more on proof-theoretic, syntactic aspects of logic, while Dedekind focuses on model-theoretic, semantic aspects. Thus, nothing like Frege's

revolutionary analysis of deductive inference, by means of his "Begriffsschrift", can be found in Dedekind. On the other hand, Dedekind is much more explicit, and clear, than Frege concerning issues such as categoricity, completeness, independence, etc., which puts him in proximity with a formal axiomatic approach as championed later by Hilbert and Bernays (Reck 2003a, Sieg & Schlimm 2005).

Compared to Frege, Dedekind also has much more to say about the infinite (not just by formulating his explicit definition of that notion, but also by exploring the possibility of different infinite cardinalities with Cantor). And he shows more awareness of the challenge posed by Kroneckerian computational and constructivist strictures to logicism. Moreover, the differences between Frege's and Dedekind's respective constructions of the natural and real numbers are noteworthy. As we saw, Dedekind conceives of the natural numbers primarily as ordinal numbers; he also identifies them purely "structurally". Frege makes their application as cardinal numbers central; and he insists on building this application into the very nature of the natural numbers, thus endowing them with non-relational, "intrinsic" properties (Reck 2003b). Similarly in the case of the real numbers.

Beyond Frege, it is instructive to compare Dedekind's approach also further with later set-theoretic ones. We noted in the last section that many of his innovations are built right into contemporary axiomatic set theory. Yet here too, significant differences emerge if we look more closely. To begin with, Dedekind does not start with an axiom of infinity, as a basic principle; instead, he tries to prove the existence of infinite sets. This can be seen as another application of the methodological rule to prove everything "capable of proof". (Dedekind's argument in this connection is also similar to an earlier one in Bolzano's posthumously published works; compare Ferreirós 1999, ch. 7.) But few set theorists today will want to defend this aspect of Dedekind's approach.

A second marked difference between Dedekind and later set theory has also already come up, but deserves further comment. This is his appeal to "abstraction" in the last step of his construction of both the natural and the real numbers ("Dedekind-abstraction", as it is called in Tait 1997). For the real numbers, the common procedure in contemporary set theory is to follow Dedekind up to this last step, but then to work with the Dedekind-cuts

E. Reck, November 2007 — Draft; please do not quote!

themselves as "the real numbers". While being well aware that this is an option (Dedekind 1876b, 1888b), Dedekind tells us to "abstract" from the cuts and to "create" additional mathematical objects determined by, but not identical with, them. Similarly for the natural numbers: Here it is now common to construct a particular simple infinity, usually using the finite von Neumann ordinals, and to identify the natural numbers with them (thus: $0 = \varnothing$, $1 = \{1\}$, $2 = \{1, 2\}$, etc.). Once again, the step involving "abstraction" and "creation" distinctive of Dedekind's procedure is avoided.

In modern set theory, it is sometimes added that any other set-theoretically constructed system isomorphic to the system of Dedekind cuts or to the system of finite von Neumann ordinals, respectively, would do as well, i.e., be usable as "the real numbers" or "the natural numbers" for all mathematical purposes. What this means is that contemporary set theory is, often implicitly and without further elaboration, supplemented by a "set-theoretic structuralist" view about the nature of mathematical objects (Reck & Price 2000). While not unrelated, this is not Dedekind's version of structuralism. (Nor does Dedekind's version coincide with other forms of structuralism prominent in contemporary philosophy of mathematics, such as those defended by Geoffrey Hellman's and Stuart Shapiro's; see again Reck & Price 2000, Reck 2003a.)

But why did mathematicians and philosophers not follow Dedekind more closely in this respect? This leads to the question: How exactly should his notions of "abstraction" and "creation" be understood, if they can be made sense of at all? Also, why did he insist on their use in the first place? Dedekind main argument in connection with the latter is the following: The real numbers should not be identified with the corresponding cuts because those cuts have "wrong properties"; namely, as sets they contain elements, something that seems "foreign" to the real numbers themselves. Similarly for the natural numbers, which should be identified "purely arithmetically", thus again as not possessing additional, set-theoretic properties (Dedekind 1876b, 1888b; see also the corresponding discussions in Tait 1997, Reck 2003a).

Dedekind's own procedure is often interpreted as follows: As suggested by his language of "abstraction" and "creation", he must be talking about psychological processes in this

connection; thus, the resulting entities—"the real number" and "the natural numbers"—must be psychological entities (existing in people's minds). But if so, his position amounts to a form of psychologism about mathematics that is deeply problematic, as Frege, Husserl, and others taught us since (Reck 2003a). Partly to avoid such a damning conclusion, partly to attribute more philosophical depth to Dedekind, one can sometimes find the following reply in the literature: While Dedekind does not say so explicitly, his psychologistic-sounding language indicates a commitment to Kantian views, in particular to Kant's transcendental psychology (Kitcher 1986, McCarty 1995). Then again, it is not obvious how that gets us around the psychologism charge (often directed against Kant too); and it still leaves open the precise form of Dedekindian "abstraction" and "creation".

A recent suggestion for how to get clearer about the form of "Dedekind abstraction" and, at the same time, avoid psychologism is this: Instead of understanding such abstraction as a psychological process, we should understand it as a logical procedure (Tait 1997, Reck 2003a). Take again the case of the natural numbers, where Dedekind is clearest about the issue (compare Sieg & Schlimm for the development of his views). What Dedekind provides us with is this: First, the language and logic to be used are specified, thus the kinds of assertions and arguments that can be made about the natural numbers; second, a particular simple infinity is constructed; third, this simple infinity is used to determine the truth values of all arithmetic sentences (by equating the truth value of such a sentence with that of the corresponding sentence for the given simple infinity); fourth, this determination is justified further by showing that all simple infinities are isomorphic (so that, if a sentence holds for one of them, it, or its translation, holds for all).

The core of the procedure so far can be summarized thus: Something holds for the natural numbers exactly when the corresponding sentence holds for all simple infinities (i.e., is a semantic consequence of the Dedekind-Peano Axioms). But what are the natural numbers then? They are the mathematical objects whose identity is determined completely by all arithmetic truths, and only by those. The guiding idea here is this: All that matters about a mathematical object, indeed all that is built into its identity and nature, is what the corresponding mathematical truths say. Thus, it is by specifying these truths that we "create", or better identify, the corresponding objects. The resulting

position is not psychologistic; and it may be called "logical structuralism" (Reck 2003a).

Interpreted along such lines, Dedekind's approach is related to that of the "American Postulate Theorists": E. Huntington, O. Veblen, etc. (Awodey & Reck 2002); there is also a close connection to, and a clear influence on, the "formal axiomatic" method employed by Hilbert and Bernays later (see also Sieg & Schlimm 2005). For all of them, what is crucial in mathematics is to establish the completeness and consistency of certain concepts, or of the corresponding systems of axioms. In Dedekind's case, completeness is to be understood in a semantic sense, as based on categoricity; likewise, consistency is to be understood semantically, as satisfiability by a system of objects (see here Dedekind 1890, as discussed in Sinaceur 1974, Reck 2003a, and Sieg & Schlimm 2005).

The syntactic exploration to these notions, especially of the notion of consistency, that later became a core part of "proof theory" is not present in Dedekind's work. As already mentioned, the proof-theoretic side of logic is underdeveloped in his work. Nor is the "finitism" characteristic of Hilbert & Bernays' later work present in Dedekind (an aspect developed in response to both the antinomies and constructivist challenges), especially not if it is understood in a metaphysical sense. Such finitism might have been acceptable for Dedekind as a methodological stance to establish consistency; but in other respects his position is strongly infinitary, as we saw. Indeed, the finite (the natural numbers) is explained in terms of the infinite (infinite sets) by him.

The way in which Dedekind's views were analyzed in this section highlights logical issues (his basic logical notions and constructions, compared to both Frege and modern axiomatic set theory) and metaphysical issues (Dedekind abstraction and creation, the structural nature of mathematical objects). There is, however, another side to Dedekind's position—another sense in which he is a logicist and structuralist—that has not yet been brought to light fully yet. That side concerns mostly methodological and epistemological issues. Some of those latter have come up briefly in connection with Dedekind's explicitly foundational work. But to shed further light on them we need to turn to his other mathematical work, starting with his contributions to algebraic number theory.

L<small>ITERATURE</small>:  The main reference point here is Reck (2003); see also Kitcher (1986), McCarty (1995), Tait (1997), Ferreirós (1999), Reck & Price (2000), Awodey & Reck (2002), Demopoulos & Clark (2005), Sieg & Schlimm (2005), Tappenden (2006).

## 6.  Algebraic Number Theory

While historians of mathematics have long emphasized the influence Dedekind's work in algebraic number theory had on the development of twentieth-century mathematics, philosophers of mathematics have only recently started to probe its significance (with a few early exceptions: Sinaceur 1979, Edwards 1980, 1983, and Stein 1988).  In order to explore that significance, we should begin by reviewing two related aspects: the roots of Dedekind's number-theoretic innovations in the works of Gauss, Dirichlet, and Kummer; the contrast of Dedekind's methodological approach in this area with that of Kronecker.

The starting point for all of the number theorists involved was the solution of various algebraic equations, especially their solution in terms of integers.  A famous example is Fermat's Last Theorem, which concerns the (non-)solubility of the equation $x^n + y^n = z^n$ by intergers, for various exponents $n$.  The approach to this problem developed by Carl Friedrich Gauss, refined by P.G.L. Dirichlet, and extended by Eduart Kummer involves studying extensions of the (field of) rational numbers, as well of the (ring of) "integers" contained in such extensions.  Thus, Gauss introduced the "Gaussian integers" ($a + bi$, where $a$ and $b$ are regular integers and $i = \sqrt{-1}$) within the complex numbers; Kummer considered more complex "cyclotonic number fields" and the "integers" in them.

What became clear along such lines is that in some such extensions the Fundamental Theorem of Arithmetic, asserting the unique factorization of all integers into powers of primes, fails.  Had such factorization been available, solutions for various equations would have been within reach.  The question became, then, whether a suitable alternative for the Fundamental Theorem could be found.  Kummer's response, obtained in a close study of some particular cases, was to introduce "ideal complex numbers" in terms of which unique factorization could be recovered.  While this move led to striking progress, the precise nature of the new mathematical objects, including the basis for their

introduction and the range of applicability of the procedure, were left unclear.

Dedekind and Kronecker knew such previous work well, including Kummer's. Both tried to justify and refine the latter. Kronecker's strategy was to consider in detail, and with the emphasis on computational or algorithmic aspects, a finite number of concrete extensions of the rational numbers, thereby extending Kummer's results step by step. Important for him was to proceed constructively and finitistically, so as to keep the mathematical phenomena under control that way (Edwards 1977). Dedekind, in contrast, approached the issue in a more abstract and set-theoretic way (Edwards 1980, 1983, Haubrich 1992, Ferreirós 1999, ch. 3). He considered algebraic number fields in general, thereby introducing the important mathematical notion of a "field" for the first time.

Dedekind also replaced Kummer's "ideal numbers" by his "ideals"—by set-theoretically constructed objects intended to play the same role with respect to unique factorization. In fact, Dedekindian ideals are infinite subsets of the algebraic number fields in question, or of the (ring of) integers contained in them, thus giving his approach again an infinitary character. (An ideal $I$ in a ring $R$ is a subset such that the sum and difference of any two elements of $I$ is again in $I$ and the product of any element of $I$ with any element of $R$ is again in $I$.) Along such lines, he was led to the introduction of other fruitful notions, such as that of "modul". A particular problem he struggled with for quite a while, in terms of finding a fully satisfactory solution, was to find a suitable extension, not only of the notion of "integer", but of "prime number" (compare Stillwell 1996, Avigad 2006).

Both Kronecker's and Dedekind's approach in algebraic number theory produced immediate results, in their own hands. Each also had a strong influence on later mathematics—Dedekind's by shaping the approaches of Hilbert, Emmie Noether, B.L. van der Waerden, Nicolas Bourbaki, etc. (Goldmann 1998, Corry 2004, McLarty 2006), Kronecker's by being picked up later by Alexander Grothendieck, among others (Edwards 1990, Reed 1995, Corfield 2003). At the same time, the contrast between Kronecker's and Dedekind's approaches provides a clear example—historically the first really significant example—of the opposition between "constructivist" and "classical" conceptions of mathematics, as they came to be called later.

Often this opposition is debated in terms of which approach is "the right one", typically with the implication that only one, but not the other, is legitimate at all. (To some degree this started with Kronecker and Dedekind themselves. We already noted that Dedekind explicitly rejected constructivist strictures on his work, although he did not reject a constructivist approach as illegitimate in general. Kronecker, with his strong opposition to the use of set-theoretic and infinitary techniques, went further in the other direction.) However, another question seems prior and more basic: How exactly should the contrast between these two approaches to algebraic number theory, and to mathematics in general, be characterized; and especially, what is its epistemological significance?

Part of the answer to the latter question is clear and has already been made explicit in our discussion so far: Dedekind's approach is set-theoretic and infinitary, while Kroneckers' is constructivist and finitary. But this leaves us with another, harder problem: What exactly is it that a set-theoretic and infinitary methodology allows us to do that Kronecker's doesn't; and vice versa? The solution to the second half of that problem is again relatively obvious: What a Kroneckerian approach provides us with is computational, algorithmic information, thus "concrete" solutions in that sense (Edwards 1980, 1990). The second half of the problem, concerning the characteristic strengths of Dedekind's approach, is what calls for further investigation. Here we don't have as clear an answer yet, especially in a fully explicit and philosophically satisfying sense.

Part of what gets in the way is that the set-theoretic and infinitary approach Dedekind championed has been so successful, and shaped twentieth-century mathematics so much, that it is hard to find the right analytic distance to it. Often all one can find, then, are platitudes, e.g., that it is "general" and "abstract"; or there are negative characterizations (provided by its opponents), such as that fails to be finitistically and constructively acceptable. Beyond that, there is the suggestion that Dedekind's approach is "structuralist". But as that term is meant now in a methodological sense, rather than the metaphysical sense clarified earlier, one still wonders what it implies. Before attempting to explicate it further, let us briefly consider Dedekind's other mathematical works, since doing so will provide us with a few further hints.

LITERATURE:  See Edwards (1977, 1980, 1983), Sinaceur (1979), Haubrich (1992), Reed (1995), Stillwell (1996), Goldmann (1998), Ferreirós (1999), Kolmogorov & Yushkevich (2001), Alten (2003), Corfield (2003), Corry (2004), Avigad (2006), McLarty (2006).

## 7.  Other Mathematical Work

Let us consider three areas in which Dedekind applied his structuralist approach to mathematics further: the theory of algebraic functions; Galois theory; and lattice theory.

Dedekind realized early on that several of the notions and techniques he had introduced in algebraic number theory could be transferred to, and fruitfully applied in, the study of algebraic functions (or algebraic function fields, as we would say now).  This realization came to fruition in "Theorie der algebraischen Funktionen einer Veränderlichen", a long, substantive article co-written with Heinrich Weber and published in 1882.  The approach taken in it led to a better understanding of Riemann surfaces, and especially, to a purely algebraic proof of the Riemann-Roch Theorem (Geyer 1982, Kolmogorov & Yushkevich 2001, ch. 2).  As these were issues of wide interest for mathematicians, the success of Dedekind's approach gave it significant legitimacy and publicity (Edwards 1983).

Second, Dedekind's approach to algebraic number theory could also be connected naturally with Evariste Galois' revolutionary approach to algebra.  In fact, Dedekind's early and systematic study of Galois theory—he was the first to lecture on them at a German university—let to a transformation of that theory itself.  In Dedekind's hands, it turned from the study of substitutions in formulas and functions invariant under such substitutions into the study of field extensions and corresponding automorphisms (Stein 1988, Alten 2003).  Moreover, Dedekind established further applications of Galois theory, e.g., in the study of modular equations and functions (Gray 2000, ch. 4).

As a third spin-off, Dedekind's number-theoretic investigations led to the investigation of the notion of a "lattice" (under the name "Dualgruppe"), at first implicitly and then explicitly.  Dedekind explored that notion further, in itself, in two later articles: "Über Zerlegungen von Zahlen durch ihre größten gemeinsamen Teiler" (1897) and "Über die

von drei Moduln erzeugte Dualgruppe" (1900). While these articles did not have the same immediate and strong impact that his other mathematical works had, they were subsequently recognized as original, systematic contributions to lattice theory, especially to the study of modular lattices (Mehrtens 1979a, ch. 2).

While none of these mathematical works can be treated in any further detail, we can make some general observations about them. Not only do they again exemplify a set-theoretic and infinitary perspective, they also display the following characteristic features: Dedekind's focus on whole systems of mathematical objects and on general laws, in such a way that results can be transferred from one case to another; his move away from particular formulas, or from particular symbolic representations, and towards more general characterizations of the underlying systems of objects, specifically in terms of functional and relational properties; more specifically, his consideration of homomorphisms, automorphisms, isomorphisms, and features invariant under such mappings; and most basically, his investigation of fundamental concepts, introduced in connection with specific cases, but later also studied in themselves.

The features just listed can, in fact, be found in all of Dedekind's writings—including his studies in algebraic number theory, on the one hand, and his foundational work on the natural numbers, the real numbers, and set theory, on the other hand. Taking them into account gives further content to calling his approach, not just "general" and "abstract", but "structuralist". It also clarifies the sense in which Dedekind's methodology was a crucial part of the emergence of "classical" or "modern" mathematics. Looking back on the development of algebra, B.L. van der Waerden later put the point as follows: "It was Evariste Galois and Richard Dedekind who gave modern algebra its structure. That's where its weight-bearing skeleton comes from" (Dedekind 1964, foreword).

LITERATURE: There is less here, especially in English; see Mehrtens (1979a), Sinaceur (1979), Geyer (1981), Scharlau (1981b), Haubrich (1992), Gray (2000), Alten (2003).

## 8. Methodology and Epistemology

In earlier sections, we surveyed Dedekind's overtly foundational writings. This led to a

characterization of the logicism and the metaphysical structuralism they embody. In the last few sections, the focus shifted to Dedekind's other mathematical works and, in particular, to the methodological structuralism they exemplify. The latter is still in need of a more systematic elaboration, however, especially its epistemological significance. Beyond just calling Dedekind's approach set-theoretic, infinitary, and non-constructive, we can now describe the methodology it exemplifies as consisting of three related parts.

The first part is closely tied to Dedekind's use of set-theoretic tools and techniques. Dedekind often uses these to construct new mathematical objects (the natural and real numbers, ideals, etc.), or even classes of such objects (algebraic number fields in general, various moduls, rings, groups, lattices, etc.). Moreover, what one can find in his writings, both in foundational and other contexts, is this: not just the introduction of infinite sets; but also their endowment with additional structural features (order relations, arithmetic or algebraic operations, etc.); and then the study of these structured systems in terms of various higher-level properties they possess (the continuity of the real number system, unique factorization in general by means of ideals, etc.).

Philosophically significant is not just that this procedure is infinitary (acceptance of actual infinities) and non-constructive (the added features do not have to be grounded in algorithms). The following should also be noted: Relational systems are investigated with respect to their overall properties, thus as structured wholes (the entire natural number sequence as an ordinal structure, etc.). And the procedure brings with it a significant extension of the proper subject matter of mathematics. Thus Dedekind leads us far beyond what is empirically or intuitively given (concrete numbers of things, geometric magnitudes, etc.)—mathematics becomes the study of relational systems much more generally (based on a general "logic of relations", as Russell will later put it).

The second important aspect of Dedekind's methodology consists of persistently (from very early on, see Dedekind 1854) trying to identify basic mathematical concepts and corresponding laws (the concepts of continuity, infinity, and simple infinity, generalized notions of whole number, prime number, the new concepts of ideal, modul, lattice, etc.). Moreover, it is important for him to find "the right definitions" in this connection. This

involves not just basic mathematical adequacy, but also desiderata such as: fruitfulness, generality, even simplicity, elegance, as well as "purity", i.e., eliminating everything that is "foreign" to a mathematical phenomenon, so that we get to its essential core. (Geometric aspects are seen as foreign to the natural and real numbers; finding a definition of prime number at just the right level of generality, and not tied to particular cases and formalisms, is another example.)

The third characteristic aspect of Dedekind's approach adds to the first two and connects them. Not only does he study entire systems or classes of systems; not only does he try to get clear about basic notions and laws. He also tends to do both, often in conjunction, by considering mappings defined on the systems studied, especially structure-preserving mapping (homomorphisms etc.), and what is invariant under them. This involves the recognition that what is crucial about a mathematical phenomenon may not lie on the surface (tied to explicit features of initial examples, particular representations, etc.), but go deeper. And while relevant deeper features are then often captured by means of set-theoretic constructions (Dedekind cuts, quotient structures, etc.), this aspect also points beyond set theory, towards category theory (Corry 2004, McLarty 2006).

These three parts, and Dedekind's structuralist approach as a whole, may not seem so extraordinary to a contemporary mathematician. But that is just testimony to how much modern mathematics has been shaped by this approach. It was certainly seen as novel, even revolutionary, at the time. Some of the negative reactions to it, by finitist and constructivist thinkers, have already been mentioned. But the extent to which Dedekind's approach diverged from what was common, at the time and before, becomes more apparent when we remember two long held assumptions overthrown by it: the view of mathematics as the science of discrete number and continuous magnitude; the view of it as having essentially to do with calculating and algorithmic reasoning. Relative to such views, Dedekind's approach involves a radical "freeing" (Stein 1988, Tait 1997).

Another way to bring to the fore the radical character of Dedekind's work is by returning to a particular innovation it contains: his definition of the infinite. Note that what he does here is to take what was widely seen as a paradoxical property (to be equinumerous with

a proper subset) as a defining characteristic (of infinite sets). And again, what he then provides is an analysis of the finite (the natural numbers) in terms of the infinite (infinite sets), also a rather bold idea. In order for innovations like these to be acceptable at all, it was important that they not only open up novel realms for mathematics, but also lead to further clarifications and results concerning older parts of mathematics—as they did in this case (the clarification of the notion of continuity, of mathematical induction, various results in algebraic number theory, in the theory of algebraic functions, etc.).

While Dedekind was a great innovator, he was, of course, not alone in leading a large part of mathematics in a set-theoretic, infinitary, and structuralist direction. In fact, he can be seen as part of a group of mathematicians—also including Dirichlet and Riemann, among others—who promoted a more "conceptual" approach in the second half of the nineteenth century (Ferreirós 1999). Of these, Dirichlet has sometimes been seen as the leader, or as the "poet's poet", including being a big influence on Dedekind (Stein 1988). As Hermann Minkowski, another major figure in this mathematical tradition, put it later (in a reflection on Dirichlet's significance on the occasion of his 100th birthday): He impressed on other mathematicians "to conquer the problems with a minimum amount of blind calculation, a maximum of clear-seeing thought" (as quoted in Stein 1988).

Riemann also had a strong influence on Dedekind, especially in two respects: first, with his explicit emphasis, primarily in his development of complex function theory, on finding simple, characteristic, and "intrinsic" concepts on which proofs were to be based, rather than relying on "extrinsic" properties connected with, say, particular formalisms or symbolisms (Mehrtens 1979b, Laugwitz 1996, Tappenden 2006); second, with his exploration of new "conceptual possibilities" (Stein 1988), including his systematic study of non-Euclidean, or "Riemannian", geometries. Another exploration of such novel possibilities can, finally, be found in the work of Dedekind's correspondent Cantor, namely Cantor's pioneering investigation of transfinite ordinal and cardinal numbers.

But can the significance of Dedekind's methodology, or of such a "conceptual" approach in general, be captured more succinctly and analyzed more deeply? One way to do so is by highlighting the methodological values embodied in it: systematicity, generality,

purity, etc. (Avigad 2006). Another is by pointing to the kind of reasoning involved, namely "conceptual" reasoning or, in terminology later made prominent by Hilbert, "axiomatic" reasoning (Sieg & Schlimm 2005). One may even talk about a novel style of reasoning as having been introduced, where "style" is not to be understood in a psychological or sociological, but in an epistemological sense. Finally, this new style might be taken to have brought with it, not just new theorems and proofs, but a distinctive kind of understanding of mathematical phenomena (Reck forthcoming).

Such attempts at analyzing the epistemological significance of Dedekind's innovations further are only recent, initial forays, clearly in need of further elaboration. But one more observation can be added already. The methodological structuralism that shapes all of Dedekind's works is not independent of the logical and metaphysical views to be found in his foundational writings. That is to say, if one adopts his general approach it is hardly possible to hold on to metaphysical views about the nature of mathematical phenomena common previously, especially to narrowly formalist, empiricist, and similar views—a structuralist epistemology, along Dedekindian lines, calls for a structuralist metaphysics. In fact, these should be seen as two sides of the same coin (Reck forthcoming, earlier also Gray 1986). Dedekind seems to have been keenly aware of this fact, even if he never spelt out his philosophical views in a direct, systematic way.

LITERATURE: Besides Mehrtens (1979b), Gray (1986), and Stein (1988), very recently see Avigad (2006), McLarty (2006), Tappenden (2006), and Reck (forthcoming).

## 9. Concluding Remarks

We started out by sketching Dedekind's contributions to the foundations of mathematics in his overtly foundational works, especially "Stetigkeit und Irrationale Zahlen" and "Was sind und was sollen die Zahlen?" We then added a survey, and preliminary analysis, of the methodological innovations in his more mainstream mathematical writings, from his work in algebraic number theory to other areas. We also noted that the logical and metaphysical position one can find in his foundational work is intimately connected with his general methodological and epistemological perspective. In this sense it is misguided

to separate his "foundational" works sharply from his "mathematical" works—he made foundationally relevant contributions throughout. In fact, the case of Dedekind is a good illustration of a more general lesson: Any strict distinction between "foundational" or "philosophical" questions about mathematics, on the one hand, and "inner-mathematical" questions, on the other, is problematic in the end, especially if one does not want to impoverish both sides (compare again Tappenden 2006, Reck forthcoming).

If we look at Dedekind's contributions to the foundations of mathematics from such a perspective, the sum total looks impressive indeed. He was not just one of the greatest mathematicians of the nineteenth and twentieth centuries, but also one of the subtlest, most insightful philosophers of mathematics. With his structuralist views about the nature of mathematical entities and the way in which to investigate them, he (together with Dirichlet, Riemann, and Cantor) was far ahead of his time. He was even, arguably, ahead of much contemporary philosophy of mathematics, especially in terms of his sensitivity to both sides. This is not to say that his position is without problems. Dedekind himself was deeply troubled about Russell's Antinomy; and the twentieth century produced additional surprises, such as Gödel's Incompleteness Theorems, which are hard to accommodate for anyone. Moreover, the methodology of mathematics has developed further since Dedekind's time, including attempts to reconcile, and integrate, "conceptual" and "computational" thinking. Then again, is there a philosophical position available today that addresses all corresponding problems and answers all relevant questions? If not, updating a Dedekindian position may be a worthwhile project.

**Dedekind's Writings (Original Texts and English Translations)**

Dedekind, Richard (1854): "Über die Einführung neuer Funktionen in der Mathematik; Habilitationsvortrag"; in Dedekind (1930-32), Vol. 3, pp. 428-438

_____ (1857): "Abriß einer Theorie der höheren Kongruenzen in bezug auf einen reellen Primzahl-Modulus"; reprinted in Dedekind (1930-32), Vol. 1, pp. 40-67

_____ (1872): "Stetigkeit und Irrationale Zahlen", Vieweg: Braunschweig; reprinted in Dedekind (1930-32), Vol. 3, pp. 315-334, and in Dedekind (1965), pp. 1-22; English trans., Dedekind (1901b)

_____ (1876a): "Bernhard Riemanns Lebenslauf", in Riemann (1876), pp. 539-558

_____ (1876b): "Briefe an Lipschitz (1-2)", in Dedekind (1930-32), Vol. 3, pp. 468-479

_____ (1877): *Sur la Théorie des Nombres Entiers Algébrique*, Gauthier-Villars; reprinted in Dedekind (1930-32), Vol. 3, pp. 262-296; English trans. Dedekind (1996)

_____ (1882): "Theorie der algebraischen Funktionen einer Veränderlichen", with H. Weber; reprinted in Dedekind (1930-32), Vol. 1, pp. 238-350

_____ (1888a): "Was sind und was sollen die Zahlen?", Vieweg: Braunschweig; reprinted in Dedekind (1930-32), Vol. 3, pp. 335-91, and in Dedekind (1965), pp. III-XI and 1-47; English trans., (Dedekind 1901c) and (revised) Dedekind (1995)

_____ (1888b): "Brief an Weber", in Dedekind (1930-32), Vol. 3, pp. 488-490

_____ (1890): "Letter to Keferstein' reprinted, in English trans., in *From Frege to Gödel*, J. v. Heijenoort, ed., Harvard University Press: Cambridge, MA, 1967, pp. 98-103

_____ (1897): "Über Zerlegungen von Zahlen durch ihre größten gemeinsamen Teiler"; reprinted in Dedekind (1930-32), Vol. II, pp. 103-147.

_____ (1900): "Über die von drei Moduln erzeugte Dualgruppe", Mathematische Annalen 53, 371-403; reprinted in Dedekind (1930-32), Vol. II, pp. 236-271.

_____ (1901a): *Essays on the Theory of Numbers*, W.W. Beman, ed. and trans., Open Court Publishing Company: Chicago, 1901; reprinted by Dover: New York, 1963; English version of Dedekind (1965)

_____ (1901b): "Continuity and Irrational Numbers", in (Dedekind 1901a), pp. 1-27; English trans. of Dedekind (1872)

_____ (1901c): "The Nature and Meaning of Numbers", in (Dedekind 1901a), pp. 29-115; English trans. of Dedekind (1888)

_____ (1930-32): *Gesammelte Mathematische Werke*, Vols. 1-3, R. Fricke, E. Noether & Ö. Ore, eds., Vieweg: Braunschweig; reprinted—except the separately published Lejeune-Dirichlet (1893)—by Chelsea Publishing Company: New York, 1969

_____ (1964): *Über die Theorie der ganzen algebraischen Zahlen*, Nachdruck des elften Supplements mit einem Geleitwort von Prof. Dr. B. L. van der Waerden, Vieweg: Braunschweig

_____ (1965): *Was sind und was sollen die Zahlen/Stetigkeit und Irrationale Zahlen. Studienausgabe*, G. Asser, ed., Vieweg: Braunschweig; earlier English version, Dedekind (1901a)

_____ (1981): "Eine Vorlesung über Algebra", in *Richard Dedekind 1831/1981. Eine Würdigung zu seinem 150. Geburtstag*, W. Scharlau, ed., Vieweg: Braunschweig, pp. 59-100

_____ (1982): "Unveröffentlichte algebraische Arbeiten Richard Dedekinds aus seiner Göttinger Zeit, 1855-1858", Archive for History of Exact Sciences 27:4, 335-367

_____ (1985): *Vorlesungen über Diffential- und Integralrechnung 1861/62*, based on notes by H. Bechtold, M.-A. Knus & W. Scharlau, eds., Vieweg: Braunschweig

\_\_\_\_\_ (1995): *What Are Numbers and What Should They Be?* H. Pogorzelski, W. Ryan & W. Snyder, eds. and trans., Research Institute for Mathematics: Orono, ME; revised English trans. of Dedekind (1888)

\_\_\_\_\_ (1996a): *Theory of Algebraic Integers*, John Stillwell, ed. and trans., Cambridge University Press; English trans. of Dedekind (1877)

\_\_\_\_\_ (1996b): "Julius Wilhelm Richard Dedekind (1831-1916)", in *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*, Vol. 2, W.B. Ewald, ed., Oxford University Press, pp. 753-837; English translation of a variety of texts by Dedekind, partly in excerpt form

Lejeune-Dirichlet, P.G. (1893): *Vorlesungen über Zahlentheorie*, fourth edition, with supplements by Richard Dedekind, Richard Dedekind, ed.; Vieweg: Braunschweig; reprinted by Chelsea: New York, 1968

\_\_\_\_\_ (1999): *Lectures on Number Theory*, with supplements by Richard Dedekind, John Stillwell, ed. and trans., American Mathematical Society: Providence; English trans. of Lejeune-Dirichlet (1893)

Lipschitz, Rudolf (1986): *Briefwechsel mit Cantor, Dedekind, Helmholtz, Kronecker, Weierstrass und anderen*, W. Scharlau, ed., Vieweg: Braunschweig

Meschkowski, H. & Nilson, W., eds. (1991): *Georg Cantor. Briefe*, Springer: Berlin; especially "Die Periode des Briefwechsels mit Dedekind — die Entstehung der Mengenlehre", pp. 29-66

Noether, E. & Cavaillès, J., eds. (1937): *Briefwechsel Cantor-Dedekind*, Hermann, Paris

Riemann, Bernard (1876): *Gesammelte Mathematische Werke und Wissenschaftlicher Nachlass*, H. Weber, ed., with assistance by R. Dedekind; 2$^{nd}$ modified ed. 1892, with a supplements added in1902, reprinted by Dover: New York, 1953

**Further Bibliography (Secondary Literature in English, French, and German)**

Alten, H.-W., et al., eds. (2003): *4000 Jahre Algebra: Geschichte, Kulturen, Menschen*, Springer: Berlin; especially Chapter 9, "Algebra an der Wende zum 20. Jahrhundert", pp. 475-550

Avigad, Jeremy (2006): "Methodology and Metaphysics in the Development of Dedekind's Theory of Ideals", in *The Architecture of Modern Mathematic*, J. Ferreiros & J. Grey, eds., Oxford University Press, pp. 159-186

Awodey, S. & Reck, E. (2002): "Categoricity and Completeness, Part I: Nineteenth-Century Axiomatics to Twentieth-Century Metalogic", History and Philosophy of Logic 23, 1-30

Bell, E.T. (1937): *Men of Mathematics*; reprinted by Simon & Schuster: New York, 1965; especially Chapter 27, "Arithmetic the Second: Kummer (1810-1893), Dedekind (1831-1916)", pp. 510-525

Belna, Jean-Pierre (1996): *La Notion de Nombre chez Dedekind, Cantor, Frege: Théories, Conceptions, et Philosophie*, Vrin: Paris

Biermann, Kurt R. (1971): "Dedekind", in *Dictionary of Scientific Biography*, Vol. 4, C.C. Gillespie, ed., pp. 1-5

Boyer, C. & Merzbach, U. (1991): *A History of Mathematics*, second ed. (revised), Wiley & Sons: New York; especially Chapter 25, "Analysis", pp. 553-574, and Chapter 26, "Algebra", pp. 575-598

Cooke, Roger (2005): "Richard Dedekind, Stetigkeit und Irrationale Zahlen (1872)", Chapter 43 of *Landmark Writings in Western Mathematics, 1640-1940*, I. Grattan-Guinness, ed., Elsevier, pp. 553-563

Corfield, David (2003): *Towards a Philosophy of Real Mathematics*, Cambridge University Press; especially Chapter 8, "Beyond the Methodology of Mathematical Research Programmes", pp. 175-203

Corry, Leo (2004): *Modern Algebra and the Rise of Mathematical Structures*, second edition (revised), Birkhäuser: Boston; especially Chapter 2: "Richard Dedekind: Numbers and Ideals", pp. 66-136

Demopoulos, W. & Clark, P. (2005): "The Logicism of Frege, Dedekind, and Russell", in *The Oxford Handbook of Philosophy of Mathematics and Logic*, S. Shapiro, ed., Oxford University Press, pp. 166-202

Dugac, Pierre (1976): *Richard Dedekind et les Fondements des Mathématiques*, Vrin: Paris

Edwards, Harold M. (1977): *Fermat's Last Theorem. A Genetic Introduction to Algebraic Number Theory*, Springer: New York

_____ (1980): "The Genesis of Ideal Theory", Archive for History of Exact Sciences 23, 321-378

_____ (1983): "Dedekind's Invention of Ideals", *Bulletin of the London Mathematical Society* 15, 8-17

Ewald, William B. (1996): "Julius Wilhelm Richard Dedekind", in *From Kant to Hilbert. A Source Book in the Foundations of Mathematics*, Vol. 2, Oxford University Press, pp. 753-754

Ferreirós, José (1993): "On the Relation between Georg Cantor and Richard Dedekind", *Historia Mathematica* 20, 343-363

_____ (1999): *Labyrinth of Thought: A History of Set Theory and its Role in Modern Mathematics*, Birkhäuser: Boston; especially: Chapter III, "Dedekind and the Set-theoretic Approach to Algebra", pp. 81-116; Chapter IV, "The Real Number System", pp. 117-144; and Chapter VII, "Sets and Maps as a Foundation for Mathematics", pp. 215-255

_____ (2005): "Richard Dedekind (1888) and Giuseppe Peano (1989), Booklets on the Foundations of Arithmetic", Chapter 47 of *Landmark Writings in Western Mathematics, 1640-1940*, I. Grattan-Guinness, ed., Elsevier, pp. 613-626

_____ (2007): "The Early Development of Set Theory", Stanford Encyclopedia of Philosophy (Spring 2007 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/entries/settheory-early//>

Gardies, Jean-Louis (1984): "Eudoxe et Dedekind", Revue d'histoire des sciences 37, 111-125

Geyer, Wulf-Dieter (1981): "Die Theorie der algebraischen Funktionen einer Veränderlichen nach Dedekind und Weber", in Scharlau (1981a), pp. 109-133

Gillies, D.A. (1982): *Frege, Dedekind, and Peano on the Foundations of Arithmetic*, Assen

Goldman, Jay R. (1998): *The Queen of Mathematics: A Historically Motivated Guide to Number Theory*, A.K. Peters: Wellesley, MA

Gray, Jeremy (1986): "The Nineteenth-Century Revolution in Mathematical Ontology", in *Revolutions in Mathematics*, D. Gillies, ed., Oxford University Press; second edition 1992, pp. 226-248

_____ (2000): *Linear Differential Equations and Group Theory, from Riemann to Poincaré*, second edition, Birkhäuser: Boston; especially Chapter IV, "Modular Equations", pp. 101-140

Haubrich, Ralf (1992): *Zur Entstehung der algebraischen Zahlentheorie Richard Dedekinds*, Ph.D. dissertation, University of Göttingen

Jahnke, Hans Niels, ed. (2003): *A History of Analysis*, American Mathematical Society; especially Chapter 1, "Antiquity", pp. 1-40, and Chapter 10: "The End of the Science of Quantity: Foundations of Analysis, 1860-1910", pp. 291-324

Jourdain, P.E.B. (1916): "Richard Dedekind (1833-1916)", *The Monist* 26, 415-427; reprinted in *Modern Logic* 3, 1993, 207-214

Kitcher, Philip (1986): "Frege, Dedekind, and the Philosophy of Mathematics", in *Frege Synthesized*, L. Haaparanta & J. Hintikka, eds., Reidel: Dordrecht, pp. 299-343

Kolmogorov, A.N. & Yushkevich, A.P. (2001): *Mathematics of the $19^{th}$ Century. Vol. I: Mathematical Logic, Algebra, Number Theory, Probability Theory*, $2^{nd}$ revised edition, Birkhäuser: Boston; especially Chapter 2, Section 3, "The Theory of Algebraic Numbers and the Beginnings of Commutative Algebra", pp. 86-135

Laugwitz, Detlef (1996): *Bernhard Riemann 1826-1866. Wendepunkte in der Auffassung der Mathematik,* Birkhäuser: Basel; English trans., *Bernhard Riemann 1826-1866. Turning Points in the Conception of Mathematics*, Birkhäuser: Boston, 1999

McCarty, David (1995): "The Mysteries of Richard Dedekind", in *From Dedekind to Gödel: Essays on the Development of the Foundations of Mathematics*, J. Hintikka, ed., Springer: New York, pp. 53-96

McLarty, Colin (2006): "Emmy Noether's 'Set Theoretic' Topology: From Dedekind to the Rise of Functors", in *The Architecture of Modern Mathematics*, J. Ferreiros & J. Grey, eds., Oxford University Press, pp. 187-208

Mehrtens, Herbert (1979a): *Entstehung der Verbandstheorie*, Gerstenberg: Hildesheim; especially Chapter 2, "Die 'Dualgruppe' von Richard Dedekind", pp. 66-126

_____ (1979b): "Das Skelett der modernen Algebra. Zur Bildung mathematischer Begriffe bei Richard Dedekind", in *Disciplinae Novae*, C.J. Sciba, ed., Vanderhoeck & Ruprecht: Göttingen, pp. 25-43

_____ (1982): "Richard Dedekind – Der Mensch und die Zahlen", Abhandlungen der Braunschweigischen Wissenschaftlichen Gesellschaft 33, 19-33

Potter, Michael (2000): *Reason's Nearest Kin: Philosophies of Arithmetic from Kant to Carnap*, Oxford University Press; especially Chapter 3, "Dedekind", pp. 81-104

Reck, Erich (2003a): "Dedekind's Structuralism: An Interpretation and Partial Defense", Synthese 137:3, 369-419

_____ (2003b): "Frege, Natural Numbers, and Arithmetic's Umbilical Cord", Manuscrito 26:2, 427-470

_____ (forthcoming): "Dedekind, Structural Reasoning, and Mathematical Understanding", in *Perspectives on Mathematical Practices, Vol. 2*, J.-P. van Bendegem & B. van Kerkhove, eds., Springer: Dordrecht

Reck, E. & Price, M. (2000): "Structures and Structuralism in Contemporary Philosophy of Mathematics", Synthese 125:3, 341-383

Reed, David (1995): *Figures of Thought*, Routledge: London; especially Chapter 4, "Number Theory in the Nineteenth Century", pp. 76-116

Scharlau, Winfried, ed. (1981a): *Richard Dedekind, 1831/1981: Eine Würdigung zu seinen 150. Geburtstag*, Vieweg: Braunschweig

_____ (1981b): "Erläuterungen zu Dedekinds Manuskript über Algebra", in Scharlau (1981a), pp. 101-108

Schlimm, Dirk (2000): *Richard Dedekind: Axiomatic Foundations of Mathematics*, M.A. thesis, Carnegie Mellon University, Pittsburgh

Sieg, W. & Schlimm, D. (2005): "Dedekind's Analysis of Number: System and Axioms", Synthese 147, 121-170

Sinaceur, M.A. (1974): "L'infini et les nombres – Commentaires de R. Dedekind à 'Zahlen' – La correspondence avec Keferstein", Revue d'histoire des sciences 27, 251-278

_____ (1979): "La méthode mathématique de Dedekind", Revue d'histoire des sciences 32, 107-142

Stein, Howard (1988): "*Logos*, Logic, and *Logistiké*: Some Philosophical Remarks on Nineteenth-Century Transformations of Mathematics", in *History and Philosophy of Modern Mathematics. Minnesota Studies in the Philosophy of Science, Vol. XI*, W. Aspray & P. Kitcher, eds., University of Minnesota Press, pp. 238-259

_____ (1990): "Eudoxos and Dedekind: On the Ancient Greek Theory of Ratios and its Relation to Modern Mathematics", *Synthese* 84, 163-211

_____ (1998): "Dedekind, Julius Wilhelm Richard", in *Routledge Encyclopedia of Philosophy*, Vol. 2, Routledge: London/New York, pp. 840-842

Stillwell, John (1996): "Translator's Introduction", in *Richard Dedekind: Theory of Algebraic Numbers*, J. Stillwell, ed., Cambridge University Press, pp. 1-50

Tait, W.W. (1996): "Frege versus Cantor and Dedekind: On the Concept of Number", in *Frege: Importance and Legacy*, M. Schirn, ed., de Gruyter: Berlin, pp. 70-113

Tappenden, Jamie (2006): "The Riemannian Background to Frege's Philosophy", in *The Architecture of Modern Mathematic*, J. Ferreiros & J. Grey, eds., Oxford University Press, pp. 97-132

Wussing, Hans (1984): *The Genesis of the Abstract Group Concept*, MIT Press; reprinted by Dover: New York, 2007


**Other Internet Resources (Online Information and Texts)**

Texts by Dedekind:

- www.ru.nl/w-en-s/gmfw/bronnen/dedekind1.html  (facsimile of "Stetigkeit und Irrationale Zahlen")

- www.ru.nl/w-en-s/gmfw/bronnen/dedekind2.html  (facsimile of "Was sind und was sollen die Zahlen?")

General Information:

- http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Dedekind.html

- http://www.britannica.com/eb/article-9029718/Richard-Dedekind

- http://en.wikipedia.org/wiki/Richard_Dedekind


**Related Entries  (Stanford Encyclopedia of Philosophy)**

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**<erich.reck@ucr.edu>**