Introduction to Metalogic

Winter, 2006

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Course Goals

Our primary goal will be to construct and explore a mathematically precise language $\mathcal{L}$.

Ultimately, $\mathcal{L}$ should be expressive enough to do mathematics in it. $\mathcal{L}$ needn't (and won't!!) be the most convenient format in which do to mathematics, but we can translate to and from it to the acceptable versions of the more convenient formats. Additionally, we should be able to provide *mathematical proofs* that $\mathcal{L}$ has the following features:

- $\mathcal{L}$ should be structurally precise, in that it does not admit of ambiguities of the sort we find in English
  - I always reward smart men and women.
  - Susan didn't feed the cat because it was sick.
- There should be a mathematically precise way of drawing inferences in $\mathcal{L}$ (i.e, of inferring a conclusion ϕ from a set of premises Γ).
  - It should avoid some of the mathematical foibles of drawing inferences in English:
    - I want to take biology in the fall.
    - Hence, I want to take biology.
- This method of drawing inferences should be "perfect":
  - If our method allows us to infer ϕ from Γ, then it should be "logically certain" that ϕ follows from Γ;
  - If it's "logically certain" that ϕ follows from Γ, then our method should allow us to infer ϕ from Γ.
- All such notions as "infer", "logically certain", etc. should be made mathematically precise.

Game Plan:

- Build a very simple language, Sentence Logic (L). L will introduce the semantic and inferential properties of the sentential connectives:
    - $(p \,\&\, q)$, $\sim p$, $(p \supset q)$, $(p \vee q)$, etc.
    - "The unknown number c is either less than 17 or it is prime."
        - $(p \vee q)$
    - "If John and Mary picked up the piano, they are both stronger than me."
        - $(j \,\&\, m) \supset (s_1 \,\&\, s_2)$
        - $t \supset (s_1 \,\&\, s_2)$
- Prove that L has the features mentioned above.
- Enrich L into $\mathcal{L}$ by adding quantifiers:
    - $\forall x(Fx \supset Gx)$, $\exists x(Hx \,\&\, Nxc)$
    - "All prime numbers greater than two are odd"

      "There is an even prime number"

      Hence(?), "There is a prime number not greater than two"

      WHAT'S LOGICALLY WRONG WITH THIS INFERENCE??
        - $\forall y((Py \,\&\, Gyc_2) \supset Oy)$
        - $\exists x(Px \,\&\, Ex)$
        - $\exists x(Px \,\&\, \sim Gxc_2)$

- Prove that $\mathcal{L}$ has the features listed above.
    - Investigate some odd and unexpected properties of $\mathcal{L}$ that follow from it having these features.