Notes on the Induction Principle

Metalogic

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Let B, A, and D be sets, and let $\mathcal{F}$ be a class of functions.

Def. A is *closed under* $\mathcal{F}$ iff for any $x_1,\dots, x_n \in$ A and any n-ary function $f \in \mathcal{F}$, we have:

$$f(x_1,\dots, x_n) \in A.$$

Def. A is *inductive on B over* $\mathcal{F}$ iff B $\subseteq$ A and A is closed under $\mathcal{F}$.

Def. D is *generated from B by* $\mathcal{F}$ iff: D = $\cap$\{X: X is inductive on B over $\mathcal{F}$\}. (Intuitively, D is generated from B by $\mathcal{F}$ when D is the “smallest” set containing B and closed under $\mathcal{F}$.)

The Induction Principle. Suppose (i) B $\subseteq$ A, (ii) A $\subseteq$ D, (iii) D is generated from B by $\mathcal{F}$, and (iv) A is closed under $\mathcal{F}$. Then A = D.

Proof. Assume (i) – (iv). By (ii), it is enough to show that D $\subseteq$ A. By (i) and (iv), A is inductive on B over $\mathcal{F}$. By (iii), D is a subset of any set that is inductive on B over $\mathcal{F}$. So D $\subseteq$ A. □

Remark. For ordinary mathematical induction, just let B = \{0\}, D = ω, and $\mathcal{F}$ = \{x+1\}.

The Induction Principle supplies us with:

A Typical Form of Induction (for Sentence Logic)

Suppose you want to show that every formula of sentence logic has some given feature P (e.g., all formulas of sentence logic have the same number of left parentheses as right parentheses). To do this:

- Let $A = \{\phi \in F_L : \phi$ has the feature $P\}$.
- Show that $S_L \subseteq A$.
  - To do this, let ϕ be an arbitrary member of S_L; show that ϕ has P.
- Show that A is closed under $\{\mathcal{E}_{\sim}, \mathcal{E}_{\supset}\}$[1].
  - To do this, let ϕ and ψ be arbitrary members of A. Now show that:
    - $(\sim\phi)$ has P;
    - $(\phi \supset \psi)$ has P.

Once you have done this, you can conclude that $A = F_L$, which is just the claim that every formula of sentence logic has the feature P.

To see why the above strategy works, just use the induction principle, where $B = S_L$, $D = F_L$, and $\mathcal{F} = \{\mathcal{E}_{\sim}, \mathcal{E}_{\supset}\}$. Then the principle states:

Suppose (i) $S_L \subseteq A$, (ii) $A \subseteq F_L$, (iii) F_L is generated from S_L by $\{\mathcal{E}_{\sim}, \mathcal{E}_{\supset}\}$, and (iv) A is closed under $\{\mathcal{E}_{\sim}, \mathcal{E}_{\supset}\}$. Then $A = F_L$.

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[1] Recall from our book that $\mathcal{E}_{\sim}$ and $\mathcal{E}_{\supset}$ are syntactic devices that create formulas out of formulas. For any given formula ϕ, $\mathcal{E}_{\sim}(\phi)$ = "$(\sim\phi)$", and for any given formulas ϕ and ψ, $\mathcal{E}_{\supset}(\phi, \psi)$ = "$(\phi \supset \psi)$".

(iii) is true by the definition of F_L (be sure you understand this point). Also, in the Typical Form of Induction given above, (ii) is trivially true. That just leaves (i) and (iv) to be proved, which is what we do in the induction argument.