Metalogic: Homework 4

Email to me by Noon, Tuesday, March 17, 2009

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Please answer four of the following questions (providing proofs for all your answers, of course).

- **Students enrolled in 105B**: Include question 5 amongst your four; I will place substantial weight on the clarity and rigor with which you construct your inductive argument.
- **Students enrolled in 205B**: Please work on this homework individually, and do not discuss it with anyone until it has been turned in.

1. We say that Th(Γ) is *λ–categorical* iff all structures whose domain is of size λ that satisfy Th(Γ) are isomorphic. We also say that Th(Γ) is *complete* iff for all formulas ϕ, Th(Γ) contains either ϕ or its negation. (a) Show that if Th(Γ) is λ-categorical, then it is complete. (b) Use (a) to show that the theory of dense linear orderings without endpoints (in a language whose only nonlogical symbol corresponds to <) is complete.

- Let $\mathcal{L}$ be a language with only finitely many nonlogical symbols. Show that the number of nonisomorphic $\mathcal{L}$-structures with finite domains is ω. [*Done in class*]

- If the term t is substitutable for x in ϕ, then

  $\models \phi^{t}_{x}[s]$ iff $\mathcal{A} \models \phi\, [s(x|\bar{s}\,(t)]$. [*Done in class*]

2. Let ϕ be the sentence $\forall x{\sim}Rxx \wedge \forall x\forall y\forall z(Rxy \supset (Ryz \supset Rxz)) \wedge \forall x\exists yRxy$ Show that if $\mathcal{A} \models \phi$, then $|\mathcal{A}|$ is infinite in size

3. Consider the following structures for a language $\mathcal{L}$

   - $\mathcal{N} = \langle \omega, g_{\mathcal{N}} \rangle$, where $P^{2\mathcal{N}} = \{\langle x,y \rangle: x < y\}$, and $f^{\mathcal{N}} = \{\langle x,y \rangle: y = x+1\}$, and $c^{\mathcal{N}} = 0$;

- $\mathcal{Z} = \langle \mathbb{Z}, g_{\mathcal{Z}} \rangle$ where $P^{2\mathcal{Z}} = \{\langle x,y \rangle : x < y\}$, and $f^{\mathcal{Z}} = \{\langle x,y \rangle : y = x+1\}$, and $c^{\mathcal{Z}} = 0$; ($\mathbb{Z}$ is the set of integers: $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.)
- $\mathcal{Q} = \langle \mathbb{Q}, g_{\mathcal{Q}} \rangle$ where $P^{2\mathcal{Q}} = \{\langle x,y \rangle : x < y\}$, and $f^{\mathcal{Q}} = \{\langle x,y \rangle : y = x+1\}$, and $c^{\mathcal{Q}} = 0$; ($\mathbb{Q}$ is the set of rational numbers: $\{\frac{m}{n} : m, n \in \mathbb{Z}\ (n \neq 0)\}$.)

Prove or disprove: For each subset of $\{\mathcal{N}, \mathcal{Q}, \mathcal{Z}\}$, there is a sentence that is true in the structures in that subset and not true in the remaining structures in $\{\mathcal{N}, \mathcal{Q}, \mathcal{Z}\}$.

4. Prove or disprove: For any set of formulas Γ there is a formula ϕ such that for any structure $\mathcal{A}$:

   $\mathcal{A} \models \Gamma$ iff $\mathcal{A} \models \phi$

   Hint: (There are many ways to solve this! You may appeal, without proof, to the statement in the following problem.)

5. Prove *by induction* the following claim: If $\mathcal{A}$ and $\mathcal{B}$ differ at most in the interpretations they assign to constants, functions and predicates that do *not* occur in ϕ, then $\mathcal{A} \models \phi\ [s]$ iff $\mathcal{B} \models \phi\ [s]$.

6. Show that for any structure $\mathcal{A}$, the number of variable assignment functions s that $\mathcal{A}$ can have can be 1 or 2^{ω}, but not ω. Can this number exceed 2^{ω} for some structure $\mathcal{A}$?

7. Let $\mathcal{Q}$ be the standard model of the rational numbers in a language with expressions that intuitively correspond to 0, <, +, and ×. Show that there exists a structure $\mathcal{A}$ such that $\{\phi: \mathcal{A} \models \phi\} = \{\phi: \mathcal{Q} \models \phi\}$, but where $\mathcal{A}$ contains "infinitesimals", i.e. positive numbers that are smaller than any positive rational number.
   - [*Hint:* For one half, add a countable infinitude of constants to the language and then work with the set: $\{\phi: \mathcal{Q} \models \phi\} \cup \{c < c_q : q \in \mathcal{Q}^+\} \cup \{0 < c\}$. Use compactness, then discuss adding an infinitesimal to a rational number.]

8. Let Mod(Γ) be the class of structures $\mathcal{A}$ such that $\mathcal{A} \models \Gamma$. For a class B of structures, let Th(B) be the set of formulas ϕ such that $\mathcal{A} \models \phi$ for all $\mathcal{A} \in B$. Now let Γ, Δ be sets of formulas, and B, C be classes of structures Show that:

(a) If $\Gamma \subseteq \Delta$, then $\mathrm{Mod}(\Delta) \subseteq \mathrm{Mod}(\Gamma)$

(b) If $B \subseteq C$, then $\mathrm{Th}(C) \subseteq \mathrm{Th}(B)$

(c) $\Gamma \subseteq \mathrm{Th}(\mathrm{Mod}(\Gamma))$

(d) $B \subseteq \mathrm{Mod}(\mathrm{Th}(B))$

Extra: Go on to prove:

(e) $\mathrm{Mod}(\Gamma) = \mathrm{Mod}(\mathrm{Th}(\mathrm{Mod}(\Gamma)))$

(f) $\mathrm{Th}(B) = \mathrm{Th}(\mathrm{Mod}(\mathrm{Th}(B)))$