Metalogic: Homework 3

Due in class on Friday, March 6, 2009

Kent Johnson

Please answer four of the following six (providing proofs for all your answers, of course). You may work on the homework questions together, but you must hand in your own assignment.

1. We say that ϕ *entails* ψ iff $\{\phi\} \models \psi$. Let ψ be a formula in which the variable x does not occur free. What is the relation between:

   $\forall x(\phi \supset \psi)$, and

   $(\exists x\phi) \supset \psi$?

   Does one entail the other? Do they both entail each other? Does neither entail the other?

2. An instance of Axiom 6 is any generalization of: $x \approx y \supset (\psi_z^x \supset \psi_z^y)$, where x and y are substitutable[1] for z in ψ. Which, if any, of the following are instances of Axiom 6? *Briefly* explain your answers.

   a. $x \approx y \supset (\forall x(Pxx \supset Pyy))$
   b. $x \approx y \supset (\forall xPxy \supset \forall xPyy)$
   c. $x \approx y \supset (\forall yPxy \supset \forall yPxy)$
   d. $x \approx y \supset (\forall xPxx \supset \forall yPyy)$

3. Prove or disprove: for all t, u $\in T_{\mathcal{L}}$, $(\phi\frac{t}{u})\frac{u}{t} = \phi$

- Show that $\vdash \forall x\forall y\forall z\forall u(x \approx y \supset (z \approx u \supset (Pxz \supset Pyu)))$ [*Done in class*]

4. Suppose we were to replace MP in our proofs with a new rule, so that the new notion of a proof is given as: $<\psi_1, \ldots, \psi_n>$ is a *proof** of ϕ from Γ iff $\psi_n = \phi$ and for all ψ_i, either:

   $\psi_i \in Ax \cup \Gamma$, or

   there are j, k < i, where $\psi_k = (\sim\psi_i \supset \sim\psi_j)$

   Show that there is a proof (in the old sense) of ϕ from Γ iff there is a proof* (in the new sense) of ϕ from Γ.

[1] In class, I used the notion of "substitutable"; the book uses the definition of "OK". Either one is fine.

[The next two problems are a little more advanced, and can be thought of as projects for those who want to look at logic in a little more depth. To get credit for them, you needn't solve them completely. Simply displaying some headway on the problems will suffice.]

5. Let K be a class of structures. Two formulas ϕ, ψ are *K-equivalent* iff for all $\mathcal{A} \in K$, $\mathcal{A} \models (\phi \equiv \psi)$. Let $\mathcal{L}$ be a language whose only nonlogical expression is: "<". Let K be the class of structures that are (countable) dense linear orderings, where "<" receives its intuitive interpretation. Let Φ be the set of formulas in L that express the claims (where x and y are any variables):

- There is a first element.
- There is a last element.
- x is the first element.
- x is the last element.
- $x < y$

Show that for any $\phi \in \mathcal{F}_{\mathcal{L}}$, there is a K-equivalent formula ψ which is built up out of formulas in Φ using only the sentential connectives ($\sim, \supset, \wedge, \vee, \equiv$). [*Extra:* There was no need to assume these structures were countable. Why?]

6. Let $\mathcal{L}$ be a language and $\mathcal{A}$ be a structure for $\mathcal{L}$. An *equivalence formula* of $\mathcal{A}$ is a formula $\phi(x_1,\ldots, x_n, y_1,\ldots, y_n)$ (with no other terms in ϕ) such that $\{(x_1,\ldots, x_n, y_1,\ldots, y_n): \phi(x_1,\ldots, x_n, y_1,\ldots, y_n)\}$ is a non-empty equivalence relation E_ϕ. (For convenience, I'll henceforth abbreviate an n-tuple such as $<a_1,\ldots a_n>$ with **a**.) Whenever $<\mathbf{a}, \mathbf{a}> \in E_\phi$, $\mathbf{a}/\phi$ denotes $<\mathbf{A}, \phi>$, where **A** is the E_ϕ-equivalence class containing **a**. Let $I_\phi = \{\mathbf{a}/\phi: \phi(\mathbf{a}, \mathbf{a})\}$. Let $\mathcal{L}^+$ be a language containing all the constants and predicates of $\mathcal{L}$, and for each n-ary function symbol $f \in \mathcal{L}$, $\mathcal{L}^+$ contains an n+1-ary relation symbol Q_f. Furthermore, for each equivalence formula ϕ of $\mathcal{L}$, $\mathcal{L}^+$ contains a one place predicate symbol P_ϕ and an n+1-ary relation symbol R_ϕ. Let $\mathcal{A}^{eq}$ be a structure where:

- $|\mathcal{A}^{eq}| = \cup\{I_\phi : \phi$ is an equivalence formula of $\mathcal{A}\}$;
- All constants and predicates of $\mathcal{L}$ are interpreted in $\mathcal{A}^{eq}$ as they were in $\mathcal{A}$, where each a $\in |\mathcal{A}|$ is identified with $a/\approx v_1v_2 \in |\mathcal{A}^{eq}|$;
- Each Q_f is interpreted as $f^{\mathcal{A}}$ (where again, each a $\in |\mathcal{A}|$ is identified with $a/\approx v_1v_2 \in |\mathcal{A}^{eq}|$);
- Each P_ϕ is interpreted as I_ϕ;
- Each R_ϕ is interpreted as the relation given by the function that maps each element of $\{\mathbf{a}: \phi(\mathbf{a}, \mathbf{a})\}$ onto $\mathbf{a}/\phi$.

Task 1: Show that $\mathcal{A}^{eq}$ "fully contains" $\mathcal{A}$ (observing our convention that a $\in |\mathcal{A}|$ is identified with $a/\approx v_1v_2 \in |\mathcal{A}^{eq}|$): there is a formula $\phi(x)$ of $\mathcal{L}^+$ such that $\{a: \mathcal{A}^{eq} \models \phi(x)[s(x|a)]\} = |\mathcal{A}|$; similarly, for each constant c, function f, and relation R of $\mathcal{L}$, there are formulas ϕ, ψ, θ such that $\{a: \mathcal{A}^{eq} \models \phi(x)[s(x|a)]\} = \{c^{\mathcal{A}}\}$, $\{\mathbf{a}: \mathcal{A}^{eq} \models \psi(\mathbf{x})[s(\mathbf{x}|\mathbf{a})]\} = f^{\mathcal{A}}$, and $\{\mathbf{a}: \mathcal{A}^{eq} \models \theta(\mathbf{x})[s(\mathbf{x}|\mathbf{a})]\} = R^{\mathcal{A}}$.

Task 2: Show that for any a $\in |\mathcal{A}^{eq}|$, there is a formula $\phi(x, \mathbf{y})$ and an n-tuple $\mathbf{b} \in |\mathcal{A}^{eq}|^n$ such that $\{a: \mathcal{A}^{eq} \models \phi(x, \mathbf{y})[s(x|a)(\mathbf{y}|\mathbf{b})]\} = \{a\}$.