Metalogic: Homework 2

Due in class on Monday, February 16, 2009

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Please supply proofs for four of the following questions. Students enrolled in 205B should include problems 1 and 6 among their four. You may work on the homework questions together, but you must hand in your own assignment.

1. Show that for any $n < \omega$, $\{p_0\} \cup \{p_{k-1} \supset p_k : 1 \leq k \leq n\} \models p_n$. Now suppose Γ contains p_0 plus an "infinitely long chain" of conditionals, i.e., $\Gamma = \{ p_0, p_0 \supset p_1, p_1 \supset p_2, \ldots, p_n \supset p_{n+1}, \ldots, p_\alpha \supset p_{\alpha+1}\}$. Use the compactness theorem to prove or disprove: $\Gamma \models p_{\alpha+1}$. [*Hint:* Γ is a poorly described set!]
   a. *Extra and utterly optional:* Use the completeness theorem to give another proof of your answer.
   b. *Extra and inconceivably unnecessary::* Use set theory to give another explanation of your answer.
2. Prove or disprove: For all ϕ and all Γ, either $\Gamma \vdash \phi$ or $\Gamma \vdash {\sim}\phi$.
3. Prove or disprove: $\Gamma \models \phi$ iff there is some finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \models \phi$.
4. Consider the "Nand" operator: $\phi|\psi$, where $v[\phi|\psi] = \mathbb{F}$ iff $v[\phi] = v[\psi] = \mathbb{T}$. Show that a language like L, but where $\{ | \}$ replaces $\{\sim, \supset\}$ is truth-functionally complete.
5. Consider the "Nor" operator: $\phi\downarrow\psi$, where $v[\phi\downarrow\psi] = \mathbb{T}$ iff $v[\phi] = v[\psi] = \mathbb{F}$. Show that a language like L, but where $\{ \downarrow \}$ replaces $\{\sim, \supset\}$ is truth-functionally complete.
6. Let M[, ,] be a three-place Boolean function that takes the truth-value assigned to the *minority* of the three component sentences. Let $\perp$ be the zero-place Boolean function that always takes the value $\mathbb{F}$. Show that a language like L, but where { M[, ,], $\perp$ } replaces $\{\sim, \supset\}$ is truth-functionally complete.
7. **For any $n \in \omega$, let f be an n-ary Boolean function. Say that f is truth-functionally complete iff a language like ours except that $\{ f \}$ replaces $\{\sim, \supset\}$ is truth-functionally complete. For a fixed arbitrary n, can you place some interesting upper or lower boundaries on the number of truth-functionally complete n-ary Boolean functions?