Metalogic: Homework 1

Due in class on Friday, January 23

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Please answer **five** of the following questions (providing proofs for all your answers, of course). You may work on the homework questions together, but you must hand in your own assignment.

1. Show that no proper initial or proper final subsequence of a formula is a formula.
2. Does Theorem 1.6 hold if we replace $\mathcal{E}_\supset$ with $\mathcal{E}'_\supset$, where $\mathcal{E}'_\supset(\phi,\psi)$ = "$\phi \supset \psi$"?
3. How about if we retain $\mathcal{E}_\supset$, and instead replace $\mathcal{E}_\sim$ with $\mathcal{E}'_\sim$, where $\mathcal{E}'_\sim(\phi)$ = "$\sim\phi$"?
4. Prove or disprove: For any $\phi \in F_L$, if every occurrence of p_6 in ϕ is replaced with an occurrence of q_7, the resulting sentence ψ is satisfiable iff ϕ is satisfiable.
5. Prove or disprove: If $\{\phi\} \models \psi$ and $\{\theta\} \models \psi$, then $\{(\sim\phi \supset \theta)\} \models \psi$
6. Where Γ and Δ are any sets of formulas of F_L, we say that $\Gamma \models \Delta$ iff $\Gamma \models \phi$, for all $\phi \in \Delta$. Prove or disprove the following statement:

   For all Γ and Δ, $\Gamma \models \Delta$ implies $\Gamma \cap \Delta \models \Delta$.
7. Let v be any truth-value assignment function. Prove that for all $\phi \in F_L$, $\bar{v}$ assigns exactly one element of $\{\mathbf{T}, \mathbf{F}\}$ to ϕ.
8. We saw earlier that our logic is truth-functionally complete: for any truth function, there is some $\phi \in F_L$ that expresses it. For a fixed arbitrary $n \in \omega$, how many n-ary truth functions are there? Be sure to explain your answer
9. Our logic has the logical connectives $\sim$ and $\supset$. Consider a logic just like ours, but with the logical connectives $\sim$ and $\equiv$ (where the latter symbol is defined as in the book). Is this new logic truth-functionally complete?
10. Can one construct formulas corresponding to two binary operations, call them $\oplus$ and $\otimes$, such that $\langle\{\mathbf{T}, \mathbf{F}\}, \oplus, \otimes\rangle$ is a field? If so, provide such formulas, and prove they form a field (what is $\mathbf{1} \oplus \mathbf{1}$, where $\mathbf{1}$ is the identity element of $\otimes$); if not, prove that this cannot happen.