Incompleteness: Homework 3

Due in class on Tuesday, June 3

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1. E is a fairly weak axiom system. It is common to explore a much more powerful system, known as Peano Arithmetic, or PA.[1] PA is produced by adding to E a countable (and decidable) set of induction axioms of the form:

$$(j(\overline{0}) \wedge \forall x(j(x) \supset j(\overline{S}x)) \supset \forall x\, j(x)$$

where $j(x)$ is any formula with one free variable x. We've seen that E $\nvdash \forall x(x \neq Sx)$. Show that, in contrast, PA $\vdash \forall x(x \neq Sx)$.

2. [*Carla Valenzuela once asked the following question*:] Does the Chinese Remainder Theorem still hold if we consider 1 and n to be relatively prime for all $n > 0$?

3. By the definability theorem, we know that since exponentiation is a recursive function, it is definable in $\mathcal{N}$: $a^b = c$ iff $\mathcal{N} \models \theta(\underline{a}, \underline{b}, \underline{c})$, for all a, b, c $\in \omega$. Construct a formula θ that satisfies this statement, but which contains no occurrences of the exponentiation symbol "E".

4. Let f $\omega^n \rightarrow \omega$ be a total function. Prove that f is recursive if f is representable in E.

5. Prove that the set $\{x: P_x[n]$ is defined for infinitely many $n \in \omega\}$ is not e.e. [*Hint:* Consider the computable function $\psi(x, y)$ = undefined, if the computation of $P_x[x]$ terminates after at most y steps, and = 1, otherwise.]

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[1] Peano Arithmetic is often given in a language without axioms or symbols for exponentiation. This restriction, it turns out, does not reduce the resulting system's expressive powers.