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On an algebraic definition of laws

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HIGHLIGHTS

- Laws of nature can be formalized as many-sorted algebras of a special type.
- Algebraic representations of the laws form a special type of groups.
- For the case when measurement outcomes are real numbers an exhaustive classification of all possible laws can be achieved.

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ABSTRACT

An algebraic definition of laws is formulated, motivated by analyzing points in Euclidean geometry and from considerations of two physical examples, Ohm's law and Newton's second law. Simple algebraic examples constructed over a field are presented.

1. Introduction

To model a *law* with algebra we need to clarify many meanings of the word *law*. We may say that a *law* is a sort of a restriction. But obviously not any restriction is a *law*. We can also say that a law is a *stable type of relation*. But what does this mean mathematically? Is it possible to develop a rule that will indicate what type of relations can be laws and what cannot?

To begin answering these questions, Kulakov (1968, 1971) proposed a mathematical theory for the concept of a law. In subsequent years this theory was developed for the case when the relations were continuously differentiable functions on smooth manifolds (Mikhailichenko, 1972). Here, these ideas are developed using an algebraic approach.

Geometry

To introduce the problem, let us consider some examples from geometry. Consider the finite set $\mathfrak{M} = \{i_1, i_2, \ldots, i_n\}$, consisting of n arbitrarily located points on a Euclidean plane. Can we say that with the arbitrary location of points there exists a particular law that relates all points of the set $\mathfrak{M}$? We have to look at all possible pairs of points of $\mathfrak{M}$ to answer this question. The number of unordered pairs is $\frac{1}{2}n(n-1)$. For each pair we use the numerical distance between them measured with a ruler to characterize their relative positions. It is assumed that measurement of the distances is exact.

Assigning the distance ℓ_{ik} to each pair of points (ik), we have a set of data obtained from the experiment which fully describes the properties of the set $\mathfrak{M}$. We can present this data set as a symmetric matrix:

	i_1	i_2	i_3	...	i_n
i_1	0	ℓ_{12}	ℓ_{13}	...	ℓ_{1n}
i_2	ℓ_{12}	0	ℓ_{23}	...	ℓ_{2n}
i_3	ℓ_{13}	ℓ_{23}	0	...	ℓ_{3n}
...	....				
i_n	ℓ_{1n}	ℓ_{2n}	ℓ_{3n}	...	0

It is clear that the distances ℓ_{ik}, ℓ_{im}, ℓ_{km} between any *three* points $i, k, m \in \mathfrak{M}$ cannot satisfy any functional dependence, because if the distances ℓ_{ik} and ℓ_{im} are fixed, the third distance ℓ_{km} can take

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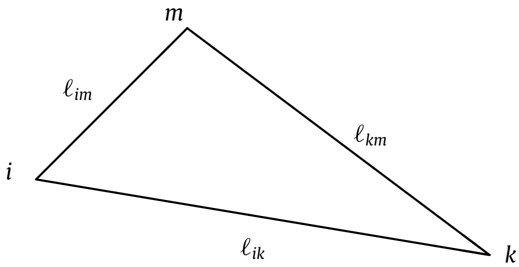
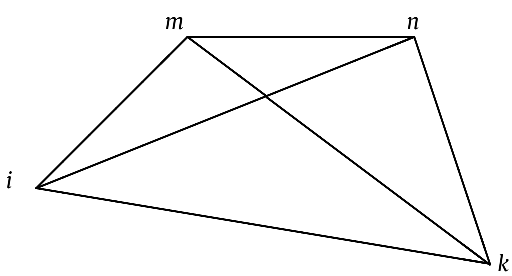
the values from $|\ell_{ik} - \ell_{im}|$ to $\ell_{ik} + \ell_{im}$.



But if we take any *four* points $i, k, m, n \in \mathfrak{M}$, then one of the *six* relative distances $\ell_{ik}, \ell_{im}, \ell_{in}, \ell_{km}, \ell_{kn}, \ell_{mn}$ is a two-valued function of the other five.



So, for every four points of the Euclidean plane there exists a functional dependence between their relative distances, which does not depend on the choice of points:

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & \ell_{ik}^2 & \ell_{im}^2 & \ell_{in}^2 \\ 1 & \ell_{ik}^2 & 0 & \ell_{km}^2 & \ell_{kn}^2 \\ 1 & \ell_{im}^2 & \ell_{km}^2 & 0 & \ell_{mn}^2 \\ 1 & \ell_{in}^2 & \ell_{kn}^2 & \ell_{mn}^2 & 0 \end{vmatrix} = 0.$$

If the four points were allowed to lie in the three-dimensional space, this determinant would be proportional to the volume of the simplex they would form. If we have zero three-dimensional volume, than all four points lie on the same plane.

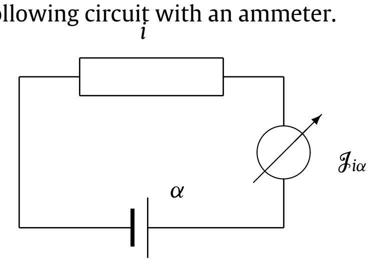
Generalizing the previous example, we can take two sets of points $i, k, m, n \in \mathfrak{M}$ and $\alpha, \beta, \gamma, \delta \in \mathfrak{M}$ of the Euclidean plane $\mathfrak{M}$ and consider the relative distances between the sets of points with Greek and Latin indexes. For any sets of points there exists a functional dependence between their relative distances, which is expressed by the Cayley–Menger determinant being zero (Kulakov, 1995).

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & \ell_{i\alpha}^2 & \ell_{i\beta}^2 & \ell_{i\gamma}^2 & \ell_{i\delta}^2 \\ 1 & \ell_{k\alpha}^2 & \ell_{k\beta}^2 & \ell_{k\gamma}^2 & \ell_{k\delta}^2 \\ 1 & \ell_{m\alpha}^2 & \ell_{m\beta}^2 & \ell_{m\gamma}^2 & \ell_{m\delta}^2 \\ 1 & \ell_{n\alpha}^2 & \ell_{n\beta}^2 & \ell_{n\gamma}^2 & \ell_{n\delta}^2 \end{vmatrix} = 0.$$

Ohm's law

In the geometry example just described, all points belong to the single set $\mathfrak{M}$. Ohm's law provides a different example where points from two different sets are matched by the result of a measurement procedure, an analog to the *distance*. (The measured values do not satisfy the triangle inequality. It is just an analogy.)

Consider the set of resistors $\mathfrak{M}$ and the set of voltage sources $\mathfrak{N}$. For any $i \in \mathfrak{M}$ and $\alpha \in \mathfrak{N}$ let us measure the electrical current in the following circuit with an ammeter.



In this case the ammeter indication $\mathcal{J}_{i\alpha}$ is an analog of the *distance* between the resistor i and the voltage source α. Consider three independent resistors $i, k, m \in \mathfrak{M}$ and two optional voltage sources $\alpha, \beta \in \mathfrak{N}$. Let us measure the *six* ammeter outputs $\mathcal{J}_{i\alpha}, \mathcal{J}_{i\beta}, \mathcal{J}_{k\alpha}, \mathcal{J}_{k\beta}, \mathcal{J}_{m\alpha}, \mathcal{J}_{m\beta}$. Assuming exact measurements, we have (Kulakov, 1968):

$$\begin{vmatrix} 1 & \mathcal{J}_{i\alpha}^{-1} & \mathcal{J}_{i\beta}^{-1} \\ 1 & \mathcal{J}_{k\alpha}^{-1} & \mathcal{J}_{k\beta}^{-1} \\ 1 & \mathcal{J}_{m\alpha}^{-1} & \mathcal{J}_{m\beta}^{-1} \end{vmatrix} = 0. \tag{1}$$

Using the reference points $k, m \in \mathfrak{M}$, $\beta \in \mathfrak{N}$, we can obtain the well-known Ohm's law for the whole circuit (Kulakov, 1968)

$$\mathcal{J}_{i\alpha} = \frac{\mathcal{E}_\alpha}{R_i + r_\alpha},$$

where $\mathcal{E}_\alpha$ is an electromotive force, r_α is the inner resistance of the voltage source α and R_i is the resistance of the resistor i.

Newton's second law

Consider Newton's second law $f = ma$, where m is the mass of the body, a is the body's acceleration and f is the driving force applied to the body. Difficulties arise when we try to understand and define the concepts of mass and force which this law contains. Mass is a measure of inertia, but this definition is implicit in the law itself. What is a force? Force – according to Lagrange – is a reason for the body's movement or a reason which intends to move. Consider the traditional statement of Newton's second law: "The driving force on a particle is equal in value and direction to the product of the material point acceleration and its mass in an inertial reference frame". Here the non-trivial concept of an *inertial reference frame* is introduced and three physical values are linked, two of which have not been defined. Is it possible to formulate Newton's second law in such a way, that does not require a definition for mass and force?

Consider two sets: the set of bodies $\mathfrak{M}$ and the set of force sources (or accelerators) $\mathfrak{N}$. One body and one force source can be paired to change the speed. We can measure such a change as acceleration $a_{i\alpha}$ of a body $i \in \mathfrak{M}$ under the applied force $\alpha \in \mathfrak{M}$.

In this case, acceleration $a_{i\alpha}$ is an analog of the *distance* between the body i and the force source α. Consider any two bodies $i, j \in \mathfrak{M}$ and any two force sources $\alpha, \beta \in \mathfrak{N}$ and measure *four* accelerations $a_{i\alpha}, a_{i\beta}, a_{j\alpha}, a_{j\beta}$. Assuming exact measurements, we have:

$$\begin{vmatrix} a_{i\alpha} & a_{i\beta} \\ a_{j\alpha} & a_{j\beta} \end{vmatrix} = 0, \tag{2}$$

by which, using the gauge points $j \in \mathfrak{M}$, $\beta \in \mathfrak{N}$, we have Newton's second law (Kulakov, 1968):

$$F_\alpha = m_i a_{i\alpha}.$$

2. Formalization

We set up the following definitions. *An algebraic system or algebra* $\langle G; \sigma \rangle$ is a set G (basic set) with the operations set σ (signature), which satisfy some axioms. *n-ary operation* f on G is $f: G^n \to G$. 0-ary operation is a selected element of G. If the algebra is defined on several sets $G_1, \ldots, G_n$, then the algebra $\langle G_1, \ldots, G_n; \sigma\rangle$ *is a many-sorted one.* A function $f: G^n \to G$ is a *partial operation* if it is defined on a subset of G^n. If partial operations $f_i \in \sigma$ are defined on the algebra $\langle G; \sigma\rangle$, then the *algebra is a partial one.*

Consider the partial many-sorted algebra $\langle \mathfrak{M}, \mathfrak{N}, B; f, g\rangle$. Functions (operations) f, g are partial ones

$$f: \mathfrak{M} \times \mathfrak{N} \to B, \qquad g: B^{n+nm+m} \to B.$$

Function f is a measurement procedure (like distance), which assigns to elements $i \in \mathfrak{M}$ and $\alpha \in \mathfrak{N}$ a value $f(i, \alpha)$ from B. Function g characterizes the relation between these values.

We say, that a given algebra defines a law of rank $(n+1, m+1)$ on subsets $\widehat{\mathfrak{M}^n} \subseteq \mathfrak{M}^n, \widehat{B^n} \subseteq B^n, \widehat{\mathfrak{N}^m} \subseteq \mathfrak{N}^m, \widehat{B^m} \subseteq B^m$ if the following axioms are satisfied:

A1. *For any tuples* $(i_1, \ldots, i_n) \in \widehat{\mathfrak{M}^n}, (b_1, \ldots, b_n) \in \widehat{B^n}$ *there exists a unique element* $\alpha \in \mathfrak{N}$, *such that:* $f(i_k, \alpha) = b_k$, *where* $k \in \{1, \ldots, n\}$.

A2. *For any tuples* $(\alpha_1, \ldots, \alpha_m) \in \widehat{\mathfrak{N}^m}, (b_1, \ldots, b_m) \in \widehat{B^m}$ *there exists a unique element* $i \in \mathfrak{M}$, *such that:* $f(i, \alpha_k) = b_k$, *where* $k \in \{1, \ldots, m\}$.

A3. *For any tuples* $(i_0, \ldots, i_n) \in \mathfrak{M} \times \widehat{\mathfrak{M}^n}, (\alpha_0, \ldots, \alpha_m) \in \mathfrak{N} \times \widehat{\mathfrak{N}^m}$ *the following is true:*

$$f(i_0, \alpha_0) = g\,(f(i_0, \alpha_1), \ldots, f(i_1, \alpha_0), \ldots, f(i_1, \alpha_1), \ldots, f(i_n, \alpha_m))\,.$$

The function g depends on all elements $f(i_j, \alpha_k) \in B$ excluding $f(i_0, \alpha_0)$.

Definition 1. The two many-sorted partial algebras $\langle \mathfrak{M}, \mathfrak{N}, B; f, g\rangle$ and $\langle \mathfrak{M}', \mathfrak{N}', B'; f', g'\rangle$ *are homomorphic if three mappings* $\lambda: \mathfrak{M} \to \mathfrak{M}'$, $\chi: \mathfrak{N} \to \mathfrak{N}'$, $\psi: B \to B'$ exist such that the following diagrams commutative

$$\begin{array}{ccc} \mathfrak{M} \times \mathfrak{N} & \xrightarrow{f} & B \\ (\lambda \times \chi)\downarrow & & \downarrow \psi, \\ \mathfrak{M}' \times \mathfrak{N}' & \xrightarrow{f'} & B' \end{array} \qquad \begin{array}{ccc} B^{mn+m+n} & \xrightarrow{g} & B \\ \psi^{mn+m+n}\downarrow & & \downarrow \psi. \\ (B')^{mn+m+n} & \xrightarrow{g'} & B' \end{array}$$

Definition 2. If the homomorphisms λ, χ, ψ are bijective, then the algebras $\langle \mathfrak{M}, \mathfrak{N}, B; f, g\rangle$, $\langle \mathfrak{M}', \mathfrak{N}', B'; f', g'\rangle$ are isomorphic or equivalent.

Definition 3. A group is an algebra $\langle B; \cdot, ^{-1}, e\rangle$ with one binary operation $(\cdot): B \times B \to B$, one unary operation $(^{-1}): B \to B$ and one nullary operation e, for which the following axioms are fulfilled:

1. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, for all $x, y, z \in B$;
2. $x \cdot e = e \cdot x = x$, for every $x \in B$;
3. $x^{-1} \cdot x = x \cdot x^{-1} = e$, for every $x \in B$.

The nullary operation selects an element $e \in B$, which is called a neutral element of the group $\langle B; \cdot, ^{-1}, e\rangle$. The unary operation $(^{-1}): B \times B \to B$ assigns to every element $x \in B$ an inverse one $x^{-1} \in B$.

If associativity is not required, but the requirement for the unique solvability of the equation $x \cdot y = z$ such that $x = z/y$ and $y = x \setminus z$ is made, then the algebra $\langle B; \cdot, /, \setminus\rangle$ is called a quasigroup. In this case the binary operations $(/): B \times B \to B$ and $(\setminus): B \times B \to B$ are called right and left divisions, respectively. They satisfy the following equations:

1. $(x \cdot y)/y = (x/y) \cdot y = x$, for all $x, y \in B$;
2. $x \setminus (x \cdot y) = x \cdot (x \setminus y) = y$, for all $x, y \in B$.

If there is a two-sided neutral element e in the quasigroup $\langle B; \cdot, /, \setminus\rangle$:

3. $(e \cdot x) = (x \cdot e) = x$, for all $x \in B$, then such an algebra $\langle B; \cdot, /, \setminus, e\rangle$ is called a loop.

Numerical representations of laws in measurement theory are constructed using the cancellation conditions (Krantz, Luce, Suppes, & Tversky, 1971, p. 354) defined on empirical systems. The cancellation conditions are considered a basic and fundamental feature of laws and are used to construct homomorphisms of empirical systems into numerical systems. In our approach, the functions f, g give the possibility of producing algebraic and numerical representations of laws without the preliminary construction of empirical systems. This approach is based on a fundamental feature of laws, the existence of functional dependence between the results of an experiment (axiom A3), which directly produces algebraic and numerical representations of laws.

3. A connection between the algebraic representation of laws of rank (2, 2) and representational measurement theory

We establish a connection between representational measurement theory and an algebraic representation of laws of rank (2, 2) by inferring the Reidemeister cancellation condition from the axioms A1–A3 for the laws of rank (2, 2).

Consider a many-sorted algebra $\langle M, N, B; f, g\rangle$ for a law of rank (2, 2) in the framework of measurement theory in the case when $\widehat{M} = M, \widehat{N} = N, \widehat{B} = B$. For the laws of rank (2, 2) the axioms A1–A3 read as follows.

Definition 4. A many-sorted algebra $\langle M, N, B; f, g\rangle$ defines a law of rank (2, 2), if the following axioms are true:

A1. $\forall i \in M,\ \forall b \in B\ \exists! \alpha \in N$ *such that* $f(i, \alpha) = b$;

A2. $\forall \alpha \in N,\ \forall b \in B\ \exists! i \in M$ *such that* $f(i, \alpha) = b$;

A3. $\forall i_0, i_1 \in M, \forall \alpha_0, \alpha_1 \in N$ we have $f(i_0, \alpha_0) = g\,(f(i_0, \alpha_1), f(i_1, \alpha_0), f(i_1, \alpha_1))$,

where $\exists!$ means unique existence.

Given a many-sorted algebra $\langle M, N, B; f, g\rangle$, satisfying Definition 4, we can derive (following Vityaev, 1985) a model $\Im = \langle M \times N; \sim\rangle$, where the equivalence relation is defined as follows:

$$(i, \alpha) \sim (j, \beta) \Leftrightarrow f(i, \alpha) = f(j, \beta).$$

In the model $\Im = \langle M \times N; \sim\rangle$ the *unrestricted solvability* axiom is true if – for every three of four elements $i, j \in M, \alpha, \beta \in N$ – there exists a fourth element such that $(i, \alpha) \sim (j, \beta)$ in Krantz et al. (1971, p. 256).

Lemma 1. *The following condition* M1 *for the model* $\Im = \langle M \times N; \sim\rangle$, *that is stronger than the axiom of unrestricted solvability, is implied by* A1, A2*:*

M1*: for every three of four elements* $i, j \in M, \alpha, \beta \in N$ *there exists a unique fourth element, such that* $(i, \alpha) \sim (j, \beta)$.

Proof. Given three elements, we choose a pair in which both elements are present, for instance (i, α). By the value $b = f(i, \alpha)$ and the third element j or β we obtain the fourth uniquely by drawing on one of the axioms A1 or A2. ■

Lemma 2. *The model* $\Im = \langle M \times N; \sim\rangle$ *satisfies the following condition* M2 *of independence (see independence relation on p.* 249 *in* Krantz et al., 1971*).*

M2*: For all* $i, j \in M, \alpha \in N$, *if* $(i, \alpha) \sim (j, \alpha)$, *then* $(i, \beta) \sim (j, \beta)$ *for every* $\beta \in N$.

Proof. By the condition M1 we have, that if $(i, \alpha) \sim (j, \alpha)$, then $i = j$ and $(i, \beta) \sim (j, \beta)$ for every $\beta \in N$. ■

Lemma 3. *The model* $\Im = \langle M \times N; \sim\rangle$ *satisfies the following Reidemeister condition (*Krantz et al., 1971, *p.* 252*):*

M3*:* $(i_0, \alpha_2) \sim (i_2, \alpha_0)$ & $(i_0, \alpha_3) \sim (i_2, \alpha_1)$ & $(i_1, \alpha_2) \sim (i_3, \alpha_0) \Rightarrow (i_1, \alpha_3) \sim (i_3, \alpha_1)$, $i_0, i_1, i_2, i_3 \in M$; $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in N$.

Proof. By replacements in A3, $i_0 \leftrightarrow i_1, \alpha_0 \to \alpha_3, \alpha_1 \to \alpha_2$, we have

$$f(i_1, \alpha_3) = g\left(f(i_1, \alpha_2),\ f(i_0, \alpha_3),\ f(i_0, \alpha_2)\right).$$

Next, by the following replacements in A3, $\alpha_1 \leftrightarrow \alpha_0$, $i_0 \to i_3$, $i_1 \to i_2$, we have $f(i_3, \alpha_1) = g(f(i_3, \alpha_0),\ f(i_2, \alpha_1),\ f(i_2, \alpha_0))$. Notice that the function g in both equations depends on equivalent values, according to the premise of condition M3. So, the function values must coincide $f(i_1, \alpha_3) = f(i_3, \alpha_1)$. ■

Lemma 4. *From the Reidemeister condition follows the hexagon condition (*Karni, 1998*)*

M4*:* $(i_0, \alpha_2) \sim (i_2, \alpha_0)$ & $(i_0, \alpha_3) \sim (i_2, \alpha_2) \sim (i_3, \alpha_0) \Rightarrow (i_2, \alpha_3) \sim (i_3, \alpha_2)$.

Proof. Let us make the following substitutions in the Reidemeister condition: $i_1 \to i_2,\ \alpha_1 \to \alpha_2$. Then we have:

$$(i_0, \alpha_2) \sim (i_2, \alpha_0) \,\&\, (i_0, \alpha_3) \sim (i_2, \alpha_2) \,\&\, (i_2, \alpha_2) \sim (i_3, \alpha_0)$$
$$\Rightarrow (i_2, \alpha_3) \sim (i_3, \alpha_2).$$

By joining two equivalences $(i_0, \alpha_3) \sim (i_2, \alpha_2)$ *and* $(i_2, \alpha_2) \sim (i_3, \alpha_0)$ into one $(i_0, \alpha_3) \sim (i_2, \alpha_2) \sim (i_3, \alpha_0)$, the hexagon condition results. ■

In measurement theory conditions M1–M4 produce numeric representations. For instance, in Karni (1998) it has been proved that if instead of relation $\sim$ we have a relation $\succsim$ on $M \times N$ which is a weak Archimedean order, satisfying the axioms of independence, unrestricted solvability, and hexagon condition, then there exist functions $\varphi : M \to Re$ $\phi : N \to Re$ such that:

$$(i, \alpha) \succsim (j, \beta) \Leftrightarrow \varphi(i) + \phi(\alpha) \geq \varphi(j) + \phi(\beta).$$

We are going to use M1–M4 not for obtaining numeric representations, but rather for obtaining an algebraic representation of the laws of rank $(2, 2)$.

Definition 5. We define an algebraic model of a law of rank $(2, 2)$ as a model $\Im = \langle M \times N; \sim\rangle$, $M \neq \emptyset$, $N \neq \emptyset$, that satisfies M1 and M3.

In Taylor (1972) it was proved using net theory that if we have the Reidemeister condition (condition M3) and two axioms whose conjunction is weaker than M1, then the model $\Im = \langle M \times N; \sim\rangle$ may be a mapping into a group.

We give another proof (which is connected with further sections and results) that a group can be defined on the model $\Im = \langle M \times N; \sim\rangle$.

We are going to analyze this model. Let $[i, \alpha]$ stand for classes of equivalent elements from $M \times N / \sim$, the set of equivalence classes we denote as $[M \times N]$.

Fix the elements $i_0 \in M,\ \alpha_0 \in N$.

Lemma 5. *The following mappings are bijections.*

$$f_{i_0} : N \to [M \times N], \qquad f_{i_0}(\alpha) = [i_0, \alpha], \quad i_0 \in M,\ \alpha \in N,$$
$$f_{\alpha_0} : M \to [M \times N], \qquad f_{\alpha_0}(i) = [i, \alpha_0], \quad i \in M,\ \alpha_0 \in N.$$

Proof. Consider function f_{i_0}. It is a one-valued correspondence because by M1, if $(i_0, \alpha) \sim (i_0, \beta)$, then $\alpha = \beta$. It is a bijection and maps the set N onto the whole set $[M \times N]$, since by M1, for all $(j, \beta) \in [j, \beta] \in [M \times N]$, $i_0 \in M$ there exists a unique $\alpha \in N$ such that $(i_0, \alpha) \sim (j, \beta)$. Bijectivity of the function f_{α_0} is proved similarly. ■

Define the inverse mappings

$$f_{\alpha_0}^{-1} : [M \times N] \to M, \qquad f_{i_0}^{-1} : [M \times N] \to N.$$

On the set $[M \times N]$ we define operation

$$[i, \alpha_0] \bullet [i_0, \alpha] = [f_{\alpha_0}^{-1}([i, \alpha_0]), f_{i_0}^{-1}([i_0, \alpha])] = [i, \alpha].$$

Lemma 6. *Operation* $\bullet$ *is left- and right-uniquely solvable and defines a quasigroup on* $[M \times N]$.

Proof. We need to prove that for any two elements $[j, \beta], [i, \alpha] \in [M \times N]$ there exist unique elements $x, y \in [M \times N]$ such that $x \bullet [j, \beta] = [i, \alpha]$, $[j, \beta] \bullet y = [i, \alpha]$. Consider the first equality. By M1, for (j, β) and i_0 there exists an element α' such that $(j, \beta) \sim (i_0, \alpha')$, and for (i, α) and α' there exists an element i' such that $(i, \alpha) \sim (i', \alpha')$. Then $x = (i', \alpha_0)$ and $[i', \alpha_0] \bullet [i_0, \alpha'] = [i', \alpha']$. The second statement can be proved in the same way. ■

The obtained quasigroup $\Im = \langle M \times N; \sim, \bullet\rangle$ is a loop, if it contains an identity element.

Lemma 7. *The element* $e = [i_0, \alpha_0]$ *is the identity element for the quasigroup* $\Im = \langle M \times N; \sim, \bullet\rangle$.

Proof. By M1 for element $[q] \in [M \times N]$ and for elements $i_0 \in M$, $\alpha_0 \in N$ there exist elements $i \in M,\ \alpha \in N$ such that $(i, \alpha_0) \sim (i_0, \alpha) \sim q$. Then $q \bullet e = [i, \alpha_0] \bullet [i_0, \alpha_0] = [i, \alpha_0] = q$, $e \bullet q = [i_0, \alpha_0] \bullet [i_0, \alpha] = [i_0, \alpha] = q$. ■

A loop is a group if it is associative.

Theorem 1. *If quasigroup* $\Im = \langle M \times N; \sim, \bullet\rangle$ *satisfies condition* M3*, then it is associative and hence a group.*

Proof. Replace all elements in condition M3 by incrementing their indices by 1.

$$(i_1, \alpha_3) \sim (i_3, \alpha_1) \,\&\, (i_1, \alpha_4) \sim (i_3, \alpha_2) \,\&\, (i_2, \alpha_3) \sim (i_4, \alpha_1)$$
$$\Rightarrow (i_2, \alpha_4) \sim (i_4, \alpha_2).$$

Let us introduce notations $p_1 = [i_1, \alpha_0]$, $p_2 = [i_2, \alpha_0]$, $p_3 = [i_3, \alpha_0]$, $p_4 = [i_4, \alpha_0]$, $q_1 = [i_0, \alpha_1]$, $q_2 = [i_0, \alpha_2]$, $q_3 = [i_0, \alpha_3]$, $q_4 = [i_0, \alpha_4]$. Then condition M3 turns into

$$p_1 \bullet q_3 = p_3 \bullet q_1 \,\&\, p_1 \bullet q_4 = p_3 \bullet q_2 \,\&\, p_2 \bullet q_3 = p_4 \bullet q_1$$
$$\Rightarrow p_2 \bullet q_4 = p_4 \bullet q_2.$$

For all x, y, z perform the following substitutions $p_1 = e$, $p_2 = x$, $p_3 = y$, $p_4 = x \bullet y$, $q_1 = e$, $q_2 = z$, $q_3 = y$, $q_4 = y \bullet z$

$$y = y \,\&\, y \bullet z = y \bullet z \,\&\, x \bullet y = x \bullet y \Rightarrow x \bullet (y \bullet z) = (x \bullet y) \bullet z.$$

Since the premise of the implication is obviously true, the conclusion is also true, which is associativity. ■

Consequence 1. *An algebraic model of a rank* $(2, 2)$ *laws is a group* $\Im = \langle M \times N; \sim, \bullet\rangle$.

4. Basic results

We have introduced the concept of a law by a many-sorted algebra $\langle \mathfrak{M}, \mathfrak{N}, B; f, g\rangle$. Consider solutions that can be obtained on the sets $\mathfrak{M}$, $\mathfrak{N}$, B and special subsets $\widehat{\mathfrak{M}^n}$, $\widehat{B^n}$, $\widehat{\mathfrak{N}^m}$, $\widehat{B^m}$.

Consider the simplest case of an algebraic system $\langle \mathfrak{M}, \mathfrak{N}, B; f, g\rangle$ by slightly modifying a theorem (Ionin, 1990). The basic meaning of the theorem is that the algebra $\langle \mathfrak{M}, \mathfrak{N}, B; f, g\rangle$ is isomorphic to the algebra $\langle B, B, B; \cdot, g\rangle$ with the same set. The proof itself is built in a sequential construction of such an isomorphic algebra.

Theorem 2 (*Ionin*). *The algebra* $\langle \mathfrak{M}, \mathfrak{N}, B; f, g\rangle$ *with* $\widehat{\mathfrak{M}} = \mathfrak{M}, \widehat{B} = B, \widehat{\mathfrak{N}} = \mathfrak{N}, \widehat{B} = B$ *is isomorphic to the algebra* $\langle B, B, B; \cdot, g\rangle$, *where* $\langle B; \cdot, ^{-1}, e\rangle$ *is a group and the mappings are*

$$f(x, u) = x \cdot u,$$
$$f(x, u) = g(f(x, v), f(y, u), f(y, v)) = f(x, v) \cdot (f(y, v))^{-1} \cdot f(y, u).$$

Proof. Indeed, by axioms A1 and A2 for any $k \in \mathfrak{M}, \gamma \in \mathfrak{N}$ we construct the mapping

$$f_k : \mathfrak{N} \to B, \qquad f_\gamma : \mathfrak{M} \to B$$

in the following way

$$f_k(\alpha) = f(k, \alpha) \qquad f_\gamma(i) = f(i, \gamma).$$

Then the triple of mappings (f_γ, f_k, id) will define the transition to the equivalent algebra $\langle B, B, B; f', g\rangle$, where

$$f'(x, y) = f(f_\gamma^{-1}(x), f_k^{-1}(y)).$$

Using any $e \in B$ and mappings $f_1(x) = f'(x, e)$, we proceed to the equivalent algebra

$$(f_1^{-1}, id, id) : \langle B, B, B; f', g\rangle \to \langle B, B, B; f'', g\rangle$$

with the mapping $f''(x, y) = f'(f_1^{-1}(x), y)$, for which the following is true:

$$f''(x, e) = f'(f_1^{-1}(x), e) = f_1(f_1^{-1}(x)) = x.$$

In the same way, using the function $f_2(x) = f''(e, x)$ we proceed to the equivalent algebra with the mapping

$$f'''(x, y) = f''(x, f_2^{-1}(y)).$$

On set B we define the operation by $x \cdot y = f'''(x, y)$. Then the following is true:

$$x \cdot e = e \cdot x = x.$$

So, by A1 and A2, algebra $\langle B; \cdot, e\rangle$ will be a loop.

For any $x, y, u, v \in B$ we have $x \cdot u = g(x \cdot v, y \cdot u, y \cdot v)$. Hence for the pairs (x, e) and $(y \cdot z, y)$ there is

$$x \cdot (y \cdot z) = g(x \cdot y, e \cdot (y \cdot z), e \cdot y) = g(x \cdot y, y \cdot z, y),$$

and on the other hand, for the pairs $(x \cdot y, y)$ and (z, e):

$$(x \cdot y) \cdot z = g((x \cdot y) \cdot e, y \cdot z, y \cdot e) = g(x \cdot y, y \cdot z, y),$$

then the operation, defined above, is associative, and hence, $\langle B; \cdot, ^{-1}, e\rangle$ is a group.

For a group we have the identity $x \cdot u = (x \cdot v) \cdot (y \cdot v)^{-1} \cdot (y \cdot u)$ such that

$$g(x, z, y) = x \cdot z^{-1} \cdot y. \quad \blacksquare \tag{3}$$

Consequence 2. *If instead of the set* B*, we consider a field* F*, then the number of non-isomorphic algebras* $\langle F, F, F; \cdot, g\rangle$ *over* F *will be the same as the number of non-isomorphic groups, constructed over* F*. If* $|F| < \infty$*, then the number of such groups depends only on the cardinality of the set* F*. If* $|F| = p$*, where* p *is a prime number, then it will be the only cyclic group (which is unique). If* $F = \mathbb{R}$ *and function* f *is continuously differentiable in all of its arguments, then, up to local isomorphism (as in the local Lie groups), it will be the only additive group (Kulakov, 1971).*

Let us check the result obtained. If we write the function (3) over the set $\mathbb{R}$ in an isomorphic multiplicative way, then we receive Eq. (2) from Newton's law.

For the consideration of rank $(n, 2)$ laws let us define the following subsets.

Definition 6. For any set B let us define a set $\triangle_{B^n}$ as follows:

$$\triangle_{B^n} = \{(x_1, \dots, x_n) \in B^n | x_1 = x_2 \text{ and/or } x_1 = x_3 \ \dots \text{ and/or } x_{n-1} = x_n\}.$$

In other words, the tuple $(x_1, \dots, x_n)$ belongs to the set $\triangle_{B^n}$, when at least two elements, belonging to this tuple, x_i and x_j for $i \neq j$ are the same: $x_i = x_j$.

Definition 7. A group $T_n(B)$ of set B transformations is called n-transitive, if for every two different tuples $(x_1, x_2, \dots, x_n) \neq (y_1, y_2, \dots, y_n) \in B^n \setminus \triangle_{B^n}$, there is $g \in T_n(B)$, for which $g(x_i) = y_i$ for $i \in \{1, \dots, n\}$. Group $T_n(B)$ of set B transformations is called sharply n-transitive, if for every two different tuples $(x_1, x_2, \dots, x_n) \neq (y_1, y_2, \dots, y_n) \in B^n \setminus \triangle_{B^n}$, there is only one element $g \in T_n(B)$, for which $g(x_i) = y_i$ for $i \in \{1, \dots, n\}$.

Definition 8. An algebraic system $\langle B, \cdot, +, ^{-1}, -, 1, 0\rangle$ is called a neardomain, if

1. $\langle B, +, -, 0\rangle$ is a loop;
2. $\langle B \setminus \{0\}, \cdot, ^{-1}, 1\rangle$ is a group;
3. $x + y = 0 \Leftrightarrow y + x = 0$;
4. $(x + y) + z = x \cdot r(y, z) + (y + z)$, where $r(y, z) \in B \setminus \{0\}$;
5. $(x + y) \cdot z = x \cdot z + y \cdot z$.

Let us consider the rank $(3, 2)$ laws with some conditions on sets $\widehat{\mathfrak{M}^2}, \widehat{F^2}, \widehat{\mathfrak{N}}, \widehat{F}$, where F is a field or neardomain.

Theorem 3. *The algebra* $\langle \mathfrak{M}, \mathfrak{N}, F; f, g\rangle$ *with* $\widehat{\mathfrak{M}^2} = \mathfrak{M}^2 \setminus \triangle_{\mathfrak{M}^2}$, $\widehat{F^2} = F^2 \setminus \triangle_{F^2}$, $\widehat{\mathfrak{N}} = \mathfrak{N}$, $\widehat{F} = F$, *is isomorphic to the algebra* $\langle F, \widehat{F^2}, F; f''', g'\rangle$ *and the mappings are*

$$f''' : F \times \widehat{F^2} \to F, \tag{4}$$

$$g'(x, y_1, y_2, z_1, z_2) = f'''\left(x, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}^{-1} \odot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right). \tag{5}$$

Such mappings define a multiplication of columns:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \odot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} f'''(x_1, y_1, y_2) \\ f'''(x_2, y_1, y_2) \end{pmatrix}. \tag{6}$$

A multiplication of columns forms a group $\langle \widehat{F^2}; \odot, ^{-1}, \binom{1}{0}\rangle$ *which is a sharply 2-transitive group.*

Proof. 1^0. For any $k_1 \neq k_2 \in \mathfrak{M}, \gamma \in \mathfrak{N}$ by A1 and A2, using the bijective mappings

$$f_1 : \mathfrak{M} \to F, \qquad F_1 : \mathfrak{N} \to \widehat{F^2},$$

we define:

$$f_1(i) = f(i, \gamma), \qquad F_1(\alpha) = (f(k_1, \alpha), f(k_2, \alpha)),$$

and we get the isomorphic algebra $\langle F, \widehat{F^2}, F; f', g\rangle$ with the function

$$f'(x, y, z) = f(f_1^{-1}(x), F_1^{-1}(y, z)).$$

Using the function

$$f_2(x) = f'(x, 1, 0),$$

where $1, 0 \in F$ and triples of mappings (f_2^{-1}, id, id) we proceed to the isomorphic algebra with the function

$$f''(x, y, z) = f'(f_2^{-1}(x), y, z).$$

Now, using another mapping

$$F_2(y, z) = (f''(1, y, z), f''(0, y, z)),$$

we get the isomorphic algebra $\langle F, \widehat{F^2}, F; f''', g'\rangle$ with the function $f'''(x, y, z) = f''(x, F_2^{-1}(y, z))$.

Notice that by the construction of this function we have:

$$f'''(1, y, z) = y, \qquad f'''(0, y, z) = z, \qquad f'''(x, 1, 0) = x. \tag{7}$$

2^0. On the set $\widehat{F^2}$ let us define an operation by (6). Then, because of A1 and A2, algebra $\langle \widehat{F^2}; \odot\rangle$ will be a quasigroup, and by (7), the algebra will be a loop. To check associativity of the defined operation, we construct a new mapping $G : \widehat{F^2}^3 \to \widehat{F^2}$ in the following way:

$$G\left(\begin{pmatrix}x_1\\x_2\end{pmatrix}, \begin{pmatrix}y_1\\y_2\end{pmatrix}, \begin{pmatrix}z_1\\z_2\end{pmatrix}\right) = \begin{pmatrix}g'(x_1, y_1, y_2, z_1, z_2)\\g'(x_2, y_1, y_2, z_1, z_2)\end{pmatrix}.$$

As in Theorem 2, we compare the mapping G, constructed, on the one hand over two tuples

$$\left(\begin{pmatrix}x_1\\x_2\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right) \qquad \left(\begin{pmatrix}y_1\\y_2\end{pmatrix} \cdot \begin{pmatrix}z_1\\z_2\end{pmatrix}, \begin{pmatrix}y_1\\y_2\end{pmatrix}\right),$$

and on the other hand, over tuples

$$\left(\begin{pmatrix}x_1\\x_2\end{pmatrix} \cdot \begin{pmatrix}y_1\\y_2\end{pmatrix}, \begin{pmatrix}y_1\\y_2\end{pmatrix}\right) \qquad \left(\begin{pmatrix}z_1\\z_2\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right),$$

we have associativity of the operation (6). Hence $\langle \widehat{F^2}; \odot, ^{-1}, \begin{pmatrix}1\\0\end{pmatrix}\rangle$ is a group.

3^0. The constructed group of transformations of the set $\widehat{F^2}$ is sharply transitive, while the same group as a group of transformations of the set F is sharply 2-transitive. It is known that a neardomain (as, in particular, near-field, skew-field, field) is linked, up to the isomorphism, to the only sharply 2-transitive group of transformations of a neardomain (Karzel, 1965, 1968).

The function G is an identity on the group $\langle \widehat{F^2}; \odot, ^{-1}, \begin{pmatrix}1\\0\end{pmatrix}\rangle$ and it is the same as the identity (3) in Theorem 2. Hence the function g' can be written in accordance with (5).

The function f''' with conditions (7) can be written (Simonov, 2006) in the following way

$$f'''(x, y, z) = x(y - z) + z,$$

which is true for neardomain F. ■

Consequence 3. *Over F the equation $g(x_2, y_1, y_2, z_1, z_2) = x_1$ can be written in an implicit form (Mikhailichenko, 1972) by setting the determinant to zero:*

$$\begin{vmatrix}x_1 & x_2 & 1\\y_1 & y_2 & 1\\z_1 & z_2 & 1\end{vmatrix} = 0.$$

If we compare the determinant from Consequence 3 with the determinant from Ohm's law, we see that Ohm's law is a law of rank (3, 2).

Let us consider an algebra $\langle \mathfrak{M}, \mathfrak{N}, \overline{F}; f, g\rangle$ connected with the laws of the rank (4, 2), where $\overline{F} = F \cup \{e_\infty\}$ and $e_\infty \notin F$.

Theorem 4. *The algebra $\langle \mathfrak{M}, \mathfrak{N}, \overline{F}; f, g\rangle$ with $\widehat{\mathfrak{M}^3} = \mathfrak{M}^3 \setminus \triangle_{\mathfrak{M}^3}$, $\widehat{\overline{F}}^3 = \overline{F}^3 \setminus \triangle_{\overline{F}^3}$, $\widehat{\mathfrak{N}} = \mathfrak{N}$, $\widehat{\overline{F}} = \overline{F}$ is isomorphic to the algebra $\langle \overline{F}, \widehat{\overline{F}}^3, \overline{F}; f''', g'\rangle$ and the mappings are*

$$f''' : F \times \widehat{F^3} \to F, \tag{8}$$

$$g'(x, y_1, y_2, y_3, z_1, z_2, z_3) = f'''\left(x, \begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}^{-1} \odot \begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}\right). \tag{9}$$

These mappings define a multiplication of columns:

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \odot \begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix} = \begin{pmatrix}f'''(x_1, y_1, y_2, y_3)\\f'''(x_2, y_1, y_2, y_3)\\f'''(x_3, y_1, y_2, y_3)\end{pmatrix}. \tag{10}$$

This multiplication of columns form a group $\langle \widehat{F^3}; \odot, ^{-1}, \begin{pmatrix}1\\0\\e_\infty\end{pmatrix}\rangle$ which is sharply 3-transitive.

Proof. Let $k_1, k_2, k_3 \in \mathfrak{M}$ be pairwise distinct elements, and $\gamma \in \mathfrak{N}$. By axioms A1 and A2 and the bijective mappings

$$f_1 : \mathfrak{M} \to \overline{F}, \qquad F_1 : \mathfrak{N} \to \widehat{\overline{F}}^3,$$

defined by:

$$f_1(i) = f(i, \gamma); \qquad F_1(\alpha) = (f(k_1, \alpha), f(k_2, \alpha), f(k_3, \alpha)),$$

we obtain an isomorphic algebra $\langle \overline{F}, \widehat{\overline{F}}^3, \overline{F}; f', g\rangle$, where function f' is given by

$$f'(x, y, z, t) = f(f_1^{-1}(x), F_1^{-1}(y, z, t)).$$

Now, using the function

$$f_2(x) = f'(x, 1, 0, e_\infty),$$

where $1, 0, e_\infty \in \overline{F}$ and the triple of mappings (f_2^{-1}, id, id), we proceed to the isomorphic algebra with function

$$f''(x, y, z, t) = f'(f_2^{-1}(x), y, z, t).$$

Then, using the mapping,

$$F_2(y, z, t) = (f''(1, y, z, t), f''(0, y, z, t), f''(e_\infty, y, z, t)),$$

we get the isomorphic algebra $\langle \overline{F}, \widehat{\overline{F}}^3, \overline{F}; f''', g\rangle$ with function

$$f'''(x, y, z, t) = f''(x, F_2^{-1}(y, z, t)).$$

Notice that by the construction of this function the following is true:

$$\begin{aligned} &f'''(1, y, z, t) = x, \qquad f'''(0, y, z, t) = z,\\ &f'''(e_\infty, y, z, t) = t, \qquad f'''(x, 1, 0, e_\infty) = x. \end{aligned} \tag{11}$$

On the set $\widehat{\overline{F}}^3$ we define an operation by (10). Then the algebra $\langle \widehat{\overline{F}}^3; \odot\rangle$, by axioms A1 and A2, is a quasigroup, and due to (11), this algebra is a loop. To check the associativity of the operation (10), as in 2^0 of Theorem 3, we construct a new mapping $G : \widehat{\overline{F}}^{3\,3} \to \widehat{\overline{F}}^3$, by which we obtain the associativity of the operation. Hence, $\langle \widehat{\overline{F}}^3; \odot, ^{-1}, \begin{pmatrix}1\\0\\e_\infty\end{pmatrix}\rangle$ is a group. This group of transformations of set $\overline{F}$ is sharply 3-transitive. ■

In order to construct sharply 3-transitive groups, Kerby and Wefelscheid (1972) defined KT-fields.

Definition 9. An algebraic system $\langle B \cup \{\infty\}; \cdot, +, ^{-1}, \varepsilon, -, 1, 0, \infty\rangle$ is a KT-field, if:

1. $\langle B; \cdot, +, ^{-1}, -, 1, 0\rangle$ is a neardomain;
2. $\varepsilon(1 - \varepsilon(x)) = 1 - \varepsilon(1 - x)$ for all $x \in B \setminus \{1, 0\}$;
3. $\varepsilon(x) \cdot \varepsilon(y) = \varepsilon(x \cdot y)$ for all $x \in B \cup \{\infty\}$, $y \in B \setminus \{0\}$.

KT-fields are linked with sharply (up to isomorphisms) 3-transitive group of transformations of the KT-field.

In particular, if the KT-field is an expanded field $\overline{F}$, then $\varepsilon(x) = x^{-1}$ is an automorphism. The corresponding sharply 3-transitive group of transformations is the group of projective transformations of the expanded field $\overline{F}$.

Consequence 4. *Any element of the group* $\widehat{\overline{F}^3}$, *for* $x_3 \neq e_\infty, 0$ *satisfying* (11), *can be written in the following way:*

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \varphi_3\left(x_1 x_3^{-1}\right) \\ \varphi_3\left(x_2 x_3^{-1}\right) \\ e_\infty \end{pmatrix} \begin{pmatrix} e_\infty \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} x_1 \\ 0 \\ e_\infty \end{pmatrix}.$$

If $x_3 = 0$, *then*

$$\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \varepsilon\left(x_1\right) \\ \varepsilon\left(x_2\right) \\ e_\infty \end{pmatrix} \begin{pmatrix} 1 \\ e_\infty \\ 0 \end{pmatrix},$$

where mappings

$\varphi_2, \varphi_3, \varepsilon : \overline{F} \to \overline{F}$

are defined as follows:

$\varphi_2(x) = f'''(x, 0, 1, e_\infty), \qquad \varepsilon(x) = f'''(x, 1, e_\infty, 0),$
$\varphi_3(x) = f'''(x, e_\infty, 0, 1),$

where f''' *is from the proof of* Theorem 4. *Then the following is true:*

$\varphi_2 \varepsilon \varphi_2(x) = \varepsilon \varphi_2 \varepsilon(x) = \varphi_3(x); \qquad \varepsilon^2(x) = x.$

The automorphism ε *of the multiplicative group extends to the whole set* $\overline{F}$ *by setting* $\varepsilon(0) = e_\infty$ *and* $\varepsilon(e_\infty) = 0$. *In this case, function* f''' *from the proof of* Theorem 4 *can be written in the following way:*

$$f'''(x, y_1, y_2, y_3) = \begin{cases} x \cdot (y_1 - y_2) + y_2, & y_3 = e_\infty \\ \varepsilon\left(x \cdot (\varepsilon(y_1) - \varepsilon(y_2)) + \varepsilon(y_2)\right), & y_3 = 0 \\ \varphi_3\left(x \cdot \left(\varphi_3\left(y_1 y_3^{-1}\right) - \varphi_3\left(y_2 y_3^{-1}\right)\right) + \varphi_3(y_2)\right) \cdot y_3, & y_3 \neq e_\infty, 0. \end{cases} \quad (12)$$

Function g can be written in the following way:

$$g(x, y_1, y_2, y_3, z_1, z_2, z_3) = f'''\left(x, \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}^{-1} \odot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right).$$

Consequence 5. *An equation* $g(x_2, y_1, y_2, y_3, z_1, z_2, z_3) = x_1$ *can be written in an implicit way (*Mikhailichenko, 1972*), by setting the following determinant zero:*

$$\begin{vmatrix} x_1 & x_2 & x_1 x_2 & 1 \\ y_1 & z_1 & y_1 z_1 & 1 \\ y_2 & z_2 & y_2 z_2 & 1 \\ y_3 & z_3 & y_3 z_3 & 1 \end{vmatrix} = 0.$$

Consequence 6. *Definition* (12) *of the function* f''' *depends on* y_3. *Let us define a new multiplicative operation, based on the standard one as follows: for two singular points the following must be true:*

$0 \cdot x = 0, \qquad e_\infty \cdot x = e_\infty,$

for the multiplication from the right we have

$x \cdot 0 = \varphi_2(x) = f'''(x, 0, 1, \infty),$
$x \cdot e_\infty = \varphi_3(x) = f'''(x, e_\infty, 0, 1)$

and

$e_\infty \cdot 0 = e_\infty, \qquad 0 \cdot e_\infty = 0$

and

$0 \cdot 0 = e_\infty \cdot e_\infty = 1.$

Thus, the operation of taking the inverse, denoted by $E : \overline{F} \to \overline{F}$, *extends to the whole set, i.e.,* $E(x) \cdot x = 1$.

If we use this multiplicative operation, then the function f''' *can be written for any* y_3:

$$f'''(x, y_1, y_2, y_3) = \varphi_3\left(\varphi_2\left(x \cdot \varphi_2\left(\varphi_3\left(y_1 \cdot y_3^{-1}\right) \times \varphi_2 E \varphi_3\left(y_2 \cdot y_3^{-1}\right)\right)\right)\right) \cdot y_3.$$

At the same time the operations of the algebraic system $\left\langle \overline{F}; \cdot, E, \varphi_2, \varphi_3, 1, 0, e_\infty \right\rangle$, will be defined on the whole set $\overline{F}$. This is in marked contrast to the partial algebraic field $\left\langle \overline{F}; \cdot, +, ^{-1}, -, 1, 0, e_\infty \right\rangle$, where operations $\left(\cdot, ^{-1}, -\right)$ are defined partially. It is easier for us to work with algebras with no such restrictions.

5. Conclusion

According to the Jordan theorem (Jordan, 1872), among the finite groups there are no sharply transitive groups with order exceeding three, except the symmetric, alternating and Mathieu groups. Among topological groups there are no sharply transitive groups with order exceeding three (Tits, 1952, 1956). For this reason, further research should consider the solutions over other sets $B, \widehat{\mathfrak{M}^n}$ and should investigate the cases $m > 1$.

The other interconnection between the algebraic representation of laws and the measurement theory appears in the results on M-point homogeneous and N-point unique groups of Alper, Narens, and Luce (see chapter 20 of Luce, Krantz, Suppes, & Tversky, 1990). In particular, Theorem 5 of Luce (2001), deals with a sharply 2-transitive group endowed with an order relation (imported from an ordered relational structure). Many-sorted algebraic systems with conditions similar to n-point uniqueness, but without an order relation, are considered for the algebraic representation of laws. Three sets of these many-sorted algebraic systems are powers of one set. Using these many-sorted algebraic systems we can construct sharply n-transitive groups for $n \leq 3$. Thus, sharply n-transitive groups appear in both approaches.

For the case when the set B is a field of real numbers $\mathbb{R}$ and functions f and g are continuously differentiable, all possible solutions were found (Mikhailichenko, 1972). These solutions (up to the locally invertible change of coordinates λ, χ (see Definition 1) in manifolds $\mathfrak{M} = \mathbb{R}^m$, $\mathfrak{N} = \mathbb{R}^n$ and scale transformation $\psi(f) \to f$) are:

for $m = n \geq 1$:

$$\begin{cases} f(i, \alpha) = x_i^1 \xi_\alpha^1 + \cdots + x_i^{m-1} \xi_\alpha^{m-1} + x_i^m \xi_\alpha^m, \\ \begin{vmatrix} f(i_1, \alpha_1) & \cdots & f(i_1, \alpha_{m+1}) \\ \vdots & \ddots & \vdots \\ f(i_{m+1}, \alpha_1) & \cdots & f(i_{m+1}, \alpha_{m+1}) \end{vmatrix} = 0, \end{cases} \quad (13)$$

$$\begin{cases} f(i, \alpha) = x_i^1 \xi_\alpha^1 + \cdots + x_i^{m-1} \xi_\alpha^{m-1} + x_i^m + \xi_\alpha^m, \\ \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & f(i_1, \alpha_1) & \cdots & f(i_1, \alpha_{m+1}) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & f(i_{m+1}, \alpha_1) & \cdots & f(i_{m+1}, \alpha_{m+1}) \end{vmatrix} = 0, \end{cases} \quad (14)$$

for $m = n - 1 \geq 2$:

$$\begin{cases} f(i, \alpha) = x_i^1 \xi_\alpha^1 + \cdots + x_i^m \xi_\alpha^m + \xi_\alpha^{m+1}, \\ \begin{vmatrix} 1 & f(i_1, \alpha_1) & \cdots & f(i_1, \alpha_{m+1}) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & f(i_{m+2}, \alpha_1) & \cdots & f(i_{m+2}, \alpha_{m+1}) \end{vmatrix} = 0, \end{cases}$$

where ψ is any continuously differentiable function of one variable, with a non-zero derivative (it is also the condition of everywhere nonzero Jacobian in the case of transformations of manifolds $\mathfrak{M}$, $\mathfrak{N}$).

So, in the case of $m = n > 1$ there are two and only two non-equivalent solutions: (13) and (14). In the case of $m = n-1 > 1$ the last solution is the only one. For all other pairs of natural numbers m and n, with $m \leq n + 2$, no solutions exist, except the only one case $(m, n) = (2, 4)$ (Theorem 4).

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