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A GENERAL THEORY OF RATIO SCALABILITY WITH
REMARKS ABOUT THE MEASUREMENT-THEORETIC
CONCEPT OF MEANINGFULNESS*

1. INTRODUCTION

The role of mathematics in empirical science is puzzling, mysterious, and in my opinion has defied rational explanation. Why mathematics should be so enormously productive and effective in the handling of empirical phenomena is clearly an important foundational problem in science, but one for the most part that has been ignored. This is somewhat surprising given the rich theory and the abundance of results that have been developed in the philosophy and foundations of mathematics. While there have been and still are great debates about the acceptability of the use of certain infinitistic and nonconstructive procedures in mathematical proofs, very little of these concerns have spilled over into science. The empirical sciences seem to have accepted without question the validity of the Platonic/Cantorian development of mathematics, and freely employ infinitistic and nonconstructive methods in their modeling of empirical phenomena. What is particularly surprising about this is that most empirically oriented scientists shy away from strong metaphysical commitments, and at the foundational level of their science usually promulgate the use of rather conservative methodologies.

There are obviously many ways of approaching the problem of explaining the role of mathematics in empirical science. The one employed in this paper, which falls under an interdisciplinary branch of science generally known as measurement theory, tries to explicate the underlying assumptions for the proper assignment of numbers to empirical variables. While this is only one aspect of justifying mathematical procedures, it is an important and necessary first step. However, depending upon one's philosophy of science and mathematics, even this restricted problem has a variety of approaches.

The approach I take is heavily colored by a vision I have of what a good theory of measurement would look like. The part that is relevant to this paper goes as follows: I see the empirical world as a finite collection of

objects with visible empirical relations holding among them. Mathematically, various parts of this situation of interest can be represented as relational structures whose domains of discourse are subsets of empirical objects. However, most of these structures that are of importance to science have a large finite number of elements, and large finite structures tend to be very complicated mathematically with measurement-theoretic properties that are usually very difficult to describe. Paradoxically, infinite structures that are idealizations of complicated finite settings are often uncomplicated and easily describable. Since most of the serious mathematics used in science is based on infinite structures, infinite idealizations are a natural step in the mathematical modeling of empirical phenomena. What is needed to justify this step are theorems that show the process of 'idealization' to be harmless, i.e., that show empirical inferences drawn from the infinite idealization are 'approximately' valid for large finite structures. While a few such results exist at this time for some types of measurement situations, this area lacks a good theoretical foundation and more research is needed. Given that infinite idealizations are wanted, the finding of appropriate idealizations becomes a problem. Here I am quite conservative: I believe that only denumerable models should be considered as idealizations of finite empirical situations. This rules out most of the idealizations that are commonly used in empirical science (e.g., ones that have dense ordering relations that are Dedekind complete). However, for many of these situations, theorems can be established that allow for the extension and preservation of measurement-theoretic properties from their natural denumerable idealization to their conventional nondenumerable idealization. An example of this type of program is carried out in Section 2 of this paper.

Conventional mathematics makes ontological commitments to the existence of infinite sets and to the validity of certain nonconstructive procedures (e.g., the Axiom of Choice). These commitments allow certain results that are otherwise not derivable to be derived which have empirical import through the conventional mathematical modeling procedures. It is my view that these kinds of results should not be taken as results about empirical phenomena but as empirical propositions that are consistent with the observations and assumptions that we make about the empirical phenomena or its idealization. I hesitate to estimate how many results of mathematical science falls into this category, but I judge it would be a significant amount. Deciding which

concepts in a mathematical model are to be treated as mathematical as opposed to empirical is often difficult. I think in certain cases concepts of meaningfulness (as developed in Section 3) might prove useful in this regard.

Empirical science often proceeds by assigning numbers to empirical entities. These assignments are called *scales* and in most applications they are ordered by the usual ordering of the real numbers. Three types of scales are in wide use: *ordinal*, *ratio*, and *interval*. These types are characterized by the permissible transformations f that can be performed on them to produce another scale of the same type: for ordinal, f is any monotonic transformation; for ratio, f is multiplication by any positive real; and for interval, f is any linear transformation of the form $f(x) = ax + b$ where a, b are reals and $a > 0$. Specifications of underlying variables of mathematical models to these scale types often impose strong conditions as to the form of the model (Luce, 1959, 1962; Falmagne and Narens, in preparation) and thus are implicitly making strong assumptions as to the nature of the empirical phenomena. Just what these assumptions are have not been very well worked out except the rather special case of ordinal scalability, where there is very little structure to preserve. In this case, the assumptions correspond to the conditions of an ordinal structure (Definition 1.1.). Special cases of ratio scalability have been worked out. The early versions of these were for the measurement of physical variables, and these axiomatizations resemble the conditions for an Archimedean ordered group. These axiomatizations have been greatly generalized and simplified over time and find their modern form in the axioms for an extensive structure (Definition 1.3.). The key axiom for this form of ratio scalability is associativity, $(x \circ y) \circ z = x \circ (y \circ z)$, and its assumption yields ratio scales in which addition has intrinsic significance. Until recently, these associative structures or their equivalents were the only qualitative models of ratio scales. The recent exception is Cohen and Narens' (1979) work on positive concatenation structures, where it is shown that for many situations the assumption of associativity in extensive structures may be deleted. In Section 2, a generalization and extension of Cohen and Narens' results is given. The research on the qualitative assumptions underlying interval scalability has been limited in the literature to models that are ultimately determined by extensive structures. However, Narens (in preparation) has a very general qualitative description of interval scalability.

The concept of invariance plays an important role in mathematics and physics. One kind of invariance that is particularly important to science is the invariance of the results of a mathematical argument under different proper numerical assignments to the empirical variables. In measurement theory, this kind of invariance is called *meaningfulness*. To date, the theory of meaningfulness is not very well developed, and it is not even clear what is the proper definition of meaningfulness. In Section 3, various concepts of meaningfulness are presented and some of their interrelationships are worked out, and the connection between these concepts and similar ones developed in geometry, especially Klein's 'Erlanger Program', are noted.

One important application of meaningfulness occurs in conjoint measurement. Luce (1978) has shown a commonly used, powerful technique of physics called *dimensional analysis* is really a meaningfulness argument for the conjoint structure of physical units. In Section 4, results concerning ratio scalability in conjoint structures are given, and the technical basis for the generalization of dimensional analysis and the structure of physical units is presented.

CONVENTION. Throughout this paper the following conventions and definitions will hold:

Re will denote the real numbers, Re^+ the positive reals, I the integers, and I^+ the positive integers.

A function $\circ: Y \times Z \rightarrow X$ is said to be a *partial (binary) operation* on X if Y and Z are subsets of X , and a *closed (binary) operation* (or just *operation*) if $Y = Z = X$. If \circ is a partial operation, then $x \circ y$ is said to be *defined* if (x, y) is in the domain of \circ , and otherwise $x \circ y$ is said to be *undefined*. As usual, $1x = x$ and if $(nx) \circ x$ is defined for some n in I^+ , then $(n + 1)x = (nx) \circ x$.

A binary relation \succsim on the set X is said to be a *weak ordering* if and only if \succsim is a transitive and connected relation. Suppose \succsim is a weak ordering on X . Define the binary relations \succ and \sim on X as follows: for each x, y in X ,

$$x \succ y \text{ if and only if } x \succsim y \text{ and not } y \succsim x,$$

and

$$x \sim y \text{ if and only if } x \succsim y \text{ and } y \succsim x.$$

Then it is easy to show that \sim is an equivalence relation on X . If in addition to being a weak ordering, \succsim is such that for each x, y in X ,

$$x \sim y \quad \text{iff} \quad x = y,$$

then \succsim is said to be a *total ordering*.

A weak ordering relation \succsim on X is said to be *dense* if and only if for each x, y in X , if $x \succ y$ then for some z in X , $x \succ z \succ y$.

$\mathcal{L} = \langle X, R_1, R_2, \dots, F_1, F_2, \dots \rangle$ is said to be a *relational structure* if and only if X is a nonempty set, R_1, R_2, \dots are relations on X , F_1 is a n_1 -ary function on X for some n_1 , F_2 is a n_2 -ary function on X for some n_2, \dots . The set X is said to be the *domain of discourse* of \mathcal{L} . Suppose \mathcal{L} is a relational structure. Then there is an appropriate language \mathcal{L} of the first order predicate calculus that describes \mathcal{L} . Any relation (or n -ary function for some n) on X that is expressible in terms of \mathcal{L} is said to be a *first order definable* relation (or function). The relations R_1, R_2, \dots and functions F_1, F_2, \dots are called the *primitives* of \mathcal{L} .

Suppose f is a function on the set X . Then $f(X) = \{f(x) | x \in X\}$. Suppose R is a n -ary relation on X . Then $f(R)$ is the n -ary relation R' on $f(X)$ such that for each x_1, \dots, x_n in X ,

$$R(x_1, \dots, x_n) \quad \text{iff} \quad R'[f(x_1), \dots, f(x_n)].$$

Suppose F is a n -ary function on X . Then $f(F)$ is the n -ary function F' on $f(X)$ such that for each x_1, \dots, x_n in X ,

$$F'[f(x_1), \dots, f(x_n)] = f[F(x_1, \dots, x_n)].$$

Suppose $\mathcal{L} = \langle X, R_1, R_2, \dots, F_1, F_2, \dots \rangle$ is a relational structure. Then $f(\mathcal{L})$ is the relational structure $\langle f(X), f(R_1), f(R_2), \dots, f(F_1), f(F_2), \dots \rangle$.

Suppose $\mathcal{L} = \langle X, R_1, R_2, \dots, F_1, F_2, \dots \rangle$ and $\mathcal{Y} = \langle Y, R'_1, R'_2, \dots, F'_1, F'_2, \dots \rangle$ are relational structures such that for $i = 1, 2, \dots$, R_i and R'_i are n_i -ary relations for some n_i , and for $j = 1, 2, \dots$, F_j and F'_j are m_j -ary functions for some m_j . Then ϕ is said to be a *homomorphism* of \mathcal{L} into \mathcal{Y} if and only if $\phi: X \rightarrow Y$ and for each x_1, \dots, x_{n_i} and each u_1, \dots, u_{m_j} in X ,

$$R_i(x_1, \dots, x_{n_i}) \quad \text{iff} \quad R'_i[\phi(x_1), \dots, \phi(x_{n_i})]$$

and

$$F_j(u_1, \dots, u_{m_j}) = F'_j[\phi(u_1), \dots, \phi(u_{m_j})].$$

ϕ is said to be an *isomorphism* of \mathcal{L} into \mathcal{Y} if and only if ϕ is a homomorphism of \mathcal{L} into \mathcal{Y} and ϕ is a one-to-one function. If ϕ is an isomorphism of

\mathcal{X} into \mathcal{Y} then $\phi(\mathcal{X})$ is said to be an *isomorphic imbedding* of \mathcal{X} into \mathcal{Y} . \mathcal{X} and \mathcal{Y} are said to be *isomorphic* if and only if there exists an isomorphism of \mathcal{X} onto \mathcal{Y} . ϕ is said to be an *endomorphism* of \mathcal{X} if and only if $\mathcal{X} = \mathcal{Y}$ and ϕ is a homomorphism of \mathcal{X} into \mathcal{Y} . ϕ is said to be an *automorphism* of \mathcal{X} if and only if ϕ is an endomorphism of \mathcal{X} , $\phi(X) = X$, and ϕ is a one-to-one function.

Suppose ϕ and ψ are arbitrary endomorphisms of $\mathcal{X} = \langle X, R_1, \dots, F_1, \dots \rangle$. Define the function $\phi * \psi$ on X as follows: for each x in X , $\phi * \psi(x) = \phi[\psi(x)]$. Then it easily follows that $\phi * \psi$ is an endomorphism of \mathcal{X} . Suppose E is the set of endomorphisms of \mathcal{X} . Then it is easy to show that $*$ is an associative operation on E . Suppose A is the set of automorphisms of \mathcal{X} . Then it is easy to show that $\langle A, * \rangle$ is a group. $\langle A, * \rangle$ is called the *group of automorphisms* of \mathcal{X} . ι will denote the identity automorphism of $\langle A, * \rangle$. For each endomorphism ϕ of \mathcal{X} , $\phi^1 = \phi$, $\phi^2 = \phi * \phi$, $\phi^3 = \phi * \phi * \phi$, etc., and if ϕ is one-to-one, ϕ^{-1} will denote the inverse of ϕ and for each n in I^+ , ϕ^{-n} will denote $(\phi^n)^{-1}$.

Throughout this paper, elements of the Cartesian product $X \times P$ will often be expressed in the form xp .

The mathematical convention of using the same symbol for the extension or restriction of an operation or relation will often be employed, e.g., as in the case of using $+$ for the addition of reals in the relational structure $\langle Re, + \rangle$ and for the addition of integers in the relational structure $\langle I, + \rangle$.

DEFINITION 1.1. $\langle X, \succsim \rangle$ is said to be an *ordinal structure* if and only if X is a nonempty set and the following two conditions hold:

- (i) \succsim is a weak ordering on X ;
- (ii) there exists a countable subset Y of X such that for each x, y in X , if $x \succ y$, then for some z in Y , $x \succ z \succ y$. \square

In theoretical work in measurement, the empirical objects (or their idealizations) are assumed to form a relational structure. This relational structure is often called the *qualitative* structure. Measurement proceeds by assigning numbers (i.e., forming scales). This process of assignment should preserve the relations of the qualitative structure, i.e., should be a homomorphism into some numerically based relational structure. Such a numerically based

structure is called the *quantitative* structure, and such homomorphisms are called *representations*.

DEFINITION 1.2. Suppose $\mathcal{X} = \langle X, \succsim \rangle$ is an ordinal structure. Then *representations* of \mathcal{X} are homomorphisms of \mathcal{X} into $\langle Re^+, \succ \rangle$. \square

The following theorem characterizes the representations of an ordinal structure:

THEOREM 1.1. *Suppose \mathcal{X} is an ordinal structure. Then (i) there exists a representation for \mathcal{X} , and (ii) for all representations ϕ, ψ of \mathcal{X} , there exists a strictly monotonic function f on Re^+ such that $\phi = f(\psi)$.*

Proof. See Theorem 2 on page 40 and Theorem 3 on page 42 of Krantz *et al.* (1971). \square

The problem with ordinal structures for measurement purposes is that they allow too many representations: As is easily verified, each strictly monotonic function from Re^+ into Re^+ applied to a representation of an ordinal structure produces another representation. The next definition characterizes an important class of structures that have a much more restrictive set of representations.

DEFINITION 1.3. Let X be a nonempty set, \succsim a binary relation on X , and \circ a partial binary operation on X . The structure $\mathcal{X} = \langle X, \succsim, \circ \rangle$ is an *extensive structure* if and only if the following eight conditions hold for all w, x, y, z in X :

- (i) *Weak ordering:* \succsim is a weak ordering.
- (ii) *Nontriviality:* there exist u, v in X such that $u \succ v$.
- (iii) *Local definability:* if $x \circ y$ is defined, $x \succsim w$, and $y \succsim z$, then $w \circ z$ is defined.
- (iv) *Monotonicity:* (1) if $x \circ z$ and $y \circ z$ are defined, then

$$x \succsim y \quad \text{iff} \quad x \circ z \succsim y \circ z,$$
 and (2) if $z \circ x$ and $z \circ y$ are defined, then

$$x \succsim y \quad \text{iff} \quad z \circ x \succsim z \circ y.$$

- (v) *Restricted solvability*: if $x \succ y$, then there exists u such that $x \succ y \circ u$.
- (vi) *Positivity*: if $x \circ y$ is defined, then $x \circ y \succ x$ and $x \circ y \succ y$.
- (vii) *Archimedean*: there exists $n \in I^+$ such that either nx is not defined or $nx \succ y$.
- (viii) *Associativity*: if $x \circ (y \circ z)$ and $(x \circ y) \circ z$ are defined, then $x \circ (y \circ z) \sim (x \circ y) \circ z$.

If \circ is an operation, then \mathcal{L} is said to be an extensive structure with a *closed* operation.

Suppose $\mathcal{L} = \langle X, \succ, \circ \rangle$ is an extensive structure. Then an *additive representation* for \mathcal{L} is a homomorphism of \mathcal{L} into $\langle Re^+, \succcurlyeq, + \rangle$. \square

THEOREM 1.2. *Suppose \mathcal{L} is an extensive structure. Then (i) an additive representation of \mathcal{L} exists, and (ii) for all additive representations ϕ, ψ of \mathcal{L} , there exists r in Re^+ such that $\phi = r\psi$.*

Proof. See Theorem 3 on page 85 of Krantz *et al.* (1971). \square

Extensive structures resemble Archimedean ordered groups. The following two well-known theorems about ordered groups are used in this paper:

THEOREM 1.3. (Hölder's Theorem). *Each Archimedean totally ordered group is isomorphic to a subgroup of $\langle Re^+, \succcurlyeq, \cdot \rangle$.* \square

THEOREM 1.4. *Each Dedekind complete, totally ordered group is isomorphic to $\langle Re^+, \succcurlyeq, \cdot \rangle$.* \square

The next definition provides an important generalization of extensive structures that will be discussed throughout the paper:

DEFINITION 1.4. \mathcal{L} is said to be a *positive concatenation structure* if and only if \mathcal{L} satisfies conditions (i) through (vii) of the definition of extensive structure (Definition 1.3.). \square

Some of the proofs of theorems of this paper rely heavily on material not presented in the paper, and thus to enhance readability, these proofs will be given at the end of the paper.

2. SCALAR STRUCTURES

CONVENTION. Throughout the rest of this section, unless explicitly stated otherwise, let

$$\mathcal{L} = \langle X, \succeq, F, R \rangle$$

where X is a nonempty set, \succeq is a total ordering on X , F is a function from $X \times X$ into X (i.e., is a binary operation on X), and R is a ternary relation on X . \square

Most of the results about \mathcal{L} in this paper naturally extend to structures of the form

$$\langle X, \succeq, F, R_1, R_2, \dots, R_i, \dots \rangle$$

where R_i is some n_i -ary relation on X . They also naturally generalize to cases where \succeq is assumed to be a weak ordering (i.e., a transitive and connected relation) rather than a total ordering. The particular case $\langle X, \succeq, F, R \rangle$ has been chosen to simplify notation throughout. Many of the results of this paper also hold in a little more general setting where F need not be a function on $X \times X$ but some relation on X .

DEFINITION 2.1. ϕ is said to be a \mathcal{N} -representation for \mathcal{L} if and only if ϕ is an isomorphic imbedding of \mathcal{L} into $\mathcal{N} = \langle N, \geq, F', R' \rangle$ where $N \subseteq \text{Re}^+$. \square

A \mathcal{N} -representation for \mathcal{L} is a quantification of the structure \mathcal{L} . The next theorem gives necessary and sufficient conditions for such a quantification to exist:

THEOREM 2.1. *The following two statements are equivalent: (i) there exists a \mathcal{N} -representation for \mathcal{L} for some \mathcal{N} ; (ii) there exists a countable subset Y of X such that for each x, z in X , if $x \succ z$, then for some y in Y , $x \succeq y \succeq z$.*

Proof. (i) implies (ii) by elementary properties of real numbers.

Assume (ii). Then by a theorem of Cantor's (as stated in Theorem 2, page 40, of Krantz *et al.* 1971), let ϕ be an isomorphic imbedding of $\langle X, \succeq \rangle$ into $\langle \text{Re}^+, \geq \rangle$. Then by letting $\mathcal{N} = \langle \phi(X), \geq, \phi(F), \phi(R) \rangle$, (i) follows. \square

Quantification of qualitative structures is a useful technique in science, since through quantification the enormous resources of mathematical analysis

can be brought to bear on empirical problems. However, for such a process to be effective, the results of a particular quantification should be independent of the particularities of its form. For our purposes, quantification consists in giving a \mathcal{N} -representation for some \mathcal{N} , and thus in specifying two entities: (i) the numerical imbedding structure, \mathcal{N} , and (ii) the particular \mathcal{N} -representation. Unfortunately, not much research effort has gone into justifying the choice of one type of numerical imbedding structure over another, although this is generally recognized as an important problem. A scattered set of proposed solutions for this type of problem for certain specific empirically generated situations exist in the literature; however, these kinds of solutions usually appeal to considerations outside of the measurement-theoretic context, e.g., to particular scientific theories that encompass a particular empirical phenomena. However, once for whatever reasons, the imbedding structure \mathcal{N} has been chosen, there seems to be great consensus about the selection of the \mathcal{N} -representation for quantification: any \mathcal{N} -representation will do. This follows from the well agreed upon rule for quantification: if the quantification based upon a particular \mathcal{N} -representation is effective to begin with, then any other \mathcal{N} -representation should produce just as an effective quantification, or to put it another way, the final results of the analysis should be independent of the particular \mathcal{N} -representation chosen. This suggests that there should be some match-up between the qualitative structure \mathcal{Z} and the numerical structure \mathcal{N} . But the nature of this match-up is not at all obvious.

The first fact to note is the obvious but important one that \mathcal{N} -representations of isomorphic structures are naturally related, i.e., that if \mathcal{Z} and \mathcal{Y} have \mathcal{N} -representations ϕ and ψ respectively and f is an isomorphism of \mathcal{Z} onto \mathcal{Y} , then ψ induces a \mathcal{N} -representation ϕ' on \mathcal{Z} by

$$\phi'(x) = \psi[f(x)].$$

Thus, if \mathcal{Z} and \mathcal{Y} are isomorphic and \mathcal{Y} has a \mathcal{N} -representation ψ , then \mathcal{Z} has a \mathcal{N} -representation that takes on exactly the same values as ψ .

The most obvious and straightforward way to make \mathcal{N} -representations equivalent is to require that they are transformable into one another, and the minimal condition for doing this is given in the following definition:

DEFINITION 2.2. Let $\mathcal{N} = \langle N, \geq, F', R' \rangle$. Then \mathcal{N} is said to be *minimally compatible* with \mathcal{Z} if and only if (i) there exists a \mathcal{N} -representation for

\mathcal{L} and (ii) for all \mathcal{N} -representations ϕ and ψ of \mathcal{L} , there exists a function g such that for all x in X ,

$$(2.1) \quad \psi(x) = g[\phi(x)]. \quad \square$$

If a \mathcal{N} -representation exists for \mathcal{L} , then \mathcal{N} is minimally compatible with \mathcal{L} since Equation (2.1) gives a method for defining a function that satisfies condition (ii) of Definition 2.2. The domain and codomain of the function g mentioned in Equation (2.1) are not specified. It follows from this equation that the domain of g must contain $\phi(X)$ and the codomain $\psi(X)$. Let g' be the restriction of g to $\phi(X)$. Then g' satisfies Equation (2.1), and for each u, v in X ,

$$\begin{aligned} g'(F'[\phi(u), \phi(v)]) &= g'(\phi[F'(u, v)]) \\ &= \psi[F'(u, v)] \\ &= F'[\psi(u), \psi(v)] \\ &= F'(g'[\phi(u)], g'[\phi(v)]), \end{aligned}$$

i.e., g' preserves the numerical function F' . Similarly, g' preserves the numerical relation R' . Thus g is an automorphism of \mathcal{N} if one imposes the additional condition that it has domain and codomain N . This latter condition seems reasonable for measurement, and it provides lots of symmetry between \mathcal{N} -representations. These considerations give rise to the following definition:

DEFINITION 2.3. \mathcal{N} is said to be *compatible* with \mathcal{L} if and only if \mathcal{N} representations for \mathcal{L} exist and for all \mathcal{N} -representations ϕ and ψ for \mathcal{L} there exists an automorphism g of \mathcal{N} such that for each x in X , $\psi(x) = g[\phi(x)]$. \square

Automorphisms will play a central role in the development presented in this section. Others (Luce, 1979, Krantz *et al.*, in preparation) consider endomorphisms to be the central concept for invariance since there are important measurement structures that have only the identity automorphism but many endomorphisms. The extensive structure $\mathcal{U} = \langle U, \geq, \oplus \rangle$, where $U = (0, 1)$ and \oplus is defined by

$$x \oplus y = z \quad \text{iff} \quad x, y, z \in U \quad \text{and} \quad x + y = z,$$

is a good example of such a structure. However, in many of the important cases in measurement theory, these structures can be extended to ones where

the endomorphisms extend to automorphisms, e.g., \mathcal{U} extends to $\langle Re^+, \geq, + \rangle$. The following theorem shows that such extensions are often possible:

THEOREM 2.2. *Suppose \mathcal{Y} is a relational structure and S is a set of one-to-one endomorphisms of \mathcal{Y} that commute with one another, i.e., $\alpha\beta = \beta\alpha$ for all α, β in S . Then there exists an extension \mathcal{Y}' of \mathcal{Y} such that each endomorphism of S extends to an automorphism of \mathcal{Y}' .*

Proof. See Theorem 5.1. \square

(The method of proof of Theorem 2.2. constructs a \mathcal{Y}' that is a substructure of an extension ${}^*\mathcal{Y}$ of \mathcal{Y} that has exactly the same relational properties as \mathcal{Y} . Thus this construction yields a \mathcal{Y}' and \mathcal{Y} that look very similar algebraically.)

An important concept in measurement that often goes under the rubric 'meaningfulness' is concerned with the invariance of certain classes of quantitative statements. As an example, let's first simplify the structure \mathcal{L} by requiring $F(x, y) = x$ and $R(x, y, z)$ holding for all x, y, z in X . This basically reduces \mathcal{L} to a totally ordered set. Let $\mathcal{N} = \langle Re^+, \geq, F', R' \rangle$ where $F'(r, s) = r$ and $R'(r, s, t)$ holds for all r, s, t in Re^+ . Then the automorphisms of \mathcal{N} are the order preserving functions from Re^+ onto Re^+ . Suppose \mathcal{L} has a \mathcal{N} -representation. Then \mathcal{L} and \mathcal{N} are compatible. Let $m(r_1, r_2, r_3, s_1, s_2, s_3)$ hold if and only if $r_1, r_2, r_3, s_1, s_2, s_3$ are positive reals and the median of $\{r_1, r_2, r_3\}$ is less than the median of $\{s_1, s_2, s_3\}$. Then for all $x_1, x_2, x_3, y_1, y_2, y_3$ in X and all \mathcal{N} -representations ϕ and ψ for \mathcal{L} ,

$$(2.2) \quad \text{iff} \quad m[\phi(x_1), \phi(x_2), \phi(x_3), \phi(y_1), \phi(y_2), \phi(y_3)] \\ m[\psi(x_1), \psi(x_2), \psi(x_3), \psi(y_1), \psi(y_2), \psi(y_3)].$$

Equation (2.2) says that comparisons of medians of \mathcal{N} -representations of triplets of elements of X are independent of the particular \mathcal{N} -representation chosen, and the relation m is said to be 'meaningful' in this measurement context. This is a quantitative form of meaningfulness that goes back to at least Stevens (1946) and has been extensively developed by Adams, Fagot, and Robinson (1965).

DEFINITION 2.4. Let $\mathcal{N} = \langle N, \geq, F', R' \rangle$. Then a n -ary relation S on N is said to be *quantitatively \mathcal{N} -meaningful* for \mathcal{L} if and only if for all \mathcal{N} -representation ϕ, ψ for \mathcal{L} ,

$$S[\phi(x_1), \dots, \phi(x_n)] \quad \text{iff} \quad S[\psi(x_1), \dots, \psi(x_n)]$$

for all x_1, \dots, x_n in X . \square

The following theorem is an immediate consequence of Definitions 2.3 and 2.4:

THEOREM 2.3. *Suppose \mathcal{L} and \mathcal{N} are compatible, $\mathcal{N} = \langle N, \geq, F', R' \rangle$, and S is a n -ary relation on N . Then if S is invariant under automorphisms of \mathcal{N} (i.e., for each r_1, \dots, r_n in N and each automorphism α of \mathcal{N} , $S[r_1, \dots, r_n]$ iff $S[\alpha(r_1), \dots, \alpha(r_n)]$), then S is quantitatively \mathcal{N} -meaningful for \mathcal{L} . \square*

Let ϕ be a \mathcal{N} -representation for \mathcal{L} . Then $\phi(\mathcal{L})$ is a substructure of \mathcal{N} , and for each automorphism α of \mathcal{L} , $\phi(\alpha)$ is an automorphism of $\phi(\mathcal{L})$, and all automorphisms of $\phi(\mathcal{L})$ are ϕ -images of automorphisms of \mathcal{L} . Suppose \mathcal{N} is compatible with \mathcal{L} . Let β be an automorphism of \mathcal{L} . Then $\psi = \phi\beta$ is a \mathcal{N} -representation of \mathcal{L} that is onto $\phi(X)$, and from this it follows that $\phi(\beta)$ is an automorphism of $\phi(\mathcal{L})$. By compatibility, let g be an automorphism of \mathcal{N} such that for each x in X ,

$$g[\phi(x)] = \psi(x) = \phi[\beta(x)],$$

i.e., g is an extension of $\phi(\beta)$. Thus each automorphism of $\phi(\mathcal{L})$ extends to an automorphism of \mathcal{N} .

Suppose $\langle X, \succ \rangle$ satisfies the following two conditions:

1. $\langle X, \succ \rangle$ has no maximal or minimal element.
2. There exists a countable subset Y of X such that for each x, z in X , if $x \succ z$ then for some y in Y , $x \succ y \succ z$.

Then it is easy to show that $\langle X, \succ \rangle$ is extendable to a Dedekind complete, totally ordered set without endpoints $\langle \bar{X}, \succeq_1 \rangle$ such that for each u, v in \bar{X} , if $u \succ_1 v$ then for some x in X , $u \succ_1 x \succ_1 v$. It also follows $\langle \bar{X}, \succeq_1 \rangle$ is unique up to isomorphism, and for this reason $\langle \bar{X}, \succeq_1 \rangle$ is called the Dedekind completion of $\langle X, \succ \rangle$. However, in general, the structure \mathcal{L} has many non-isomorphic Dedekind completions since any relational structure of the form $\bar{\mathcal{L}} = \langle \bar{X}, \succeq_1, \bar{F}, \bar{R} \rangle$, where \bar{F} and \bar{R} are any extensions of F and R respectively, will do for a 'Dedekind completion' of \mathcal{L} . In measurement, one often

takes representations of \mathcal{L} into a Dedekind complete numerical structure, $\mathcal{N} = \langle Re^+, \geq, F', R' \rangle$, and since a \mathcal{N} -representation of \mathcal{L} is an isomorphic imbedding of \mathcal{L} , these numerical representing structures are in essence Dedekind completions of \mathcal{L} . (And similarly, all Dedekind completions of \mathcal{L} are isomorphic to some Dedekind complete numerical structure since in the current context, $\langle \bar{X}, \succeq_1 \rangle$ has what G. Cantor called 'order type θ ', and thus by a well-known theorem of his has an isomorphism h onto $\langle Re^+, \geq \rangle$, and thus h is a representation of $\bar{\mathcal{L}}$ onto $\langle Re^+, \geq, h(\bar{F}), h(\bar{R}) \rangle$.) The best case for measurement is where various measurement considerations lead to a unique (up to isomorphism) Dedekind completion of \mathcal{L} ; the other case, having many nonisomorphic Dedekind completions, greatly compounds – and perhaps makes insolvable – the problem of finding the appropriate numerical representing structure. One natural condition for limiting the set of Dedekind completions of \mathcal{L} is to interpret the compatibility condition for representations (Definition 2.3.) solely in terms of the structure \mathcal{L} , i.e., to require that Dedekind completions $\bar{\mathcal{L}}$ of \mathcal{L} satisfy the following condition: each automorphism α of \mathcal{L} extends to an automorphism $\bar{\alpha}$ of $\bar{\mathcal{L}}$. Unfortunately, this condition by itself is in general not enough to insure the existence of a unique Dedekind completion.

Ratio scalings of \mathcal{L} , if they exist, are generally into structures of the form $\mathcal{N} = \langle N, \geq, F', R' \rangle$ where $N \subseteq Re^+$. Since for such situations, $r\phi$ is a representation for each \mathcal{N} -representation ϕ and each r in Re^+ , N must be Re^+ , that is, \mathcal{N} must be Dedekind complete. Some of the following definitions will give additional conditions on \mathcal{L} sufficient for the existence of ratio scales for \mathcal{L} , and some of the following theorems will show that for structures satisfying these conditions, the best possible results in terms of the above discussion also apply for extensions of automorphisms and Dedekind completions of \mathcal{L} .

DEFINITION 2.5. \mathcal{L} is said to be a *scalar structure* if and only if the following three conditions hold:

- (i) *Density*: for each x, y in X , if $x \succ y$ then for some z in X , $x \succ z \succ y$:

- (ii) *Homogeneity*: for each x, y in X , there is an automorphism α of \mathcal{L} such that $\alpha(x) = y$;
- (iii) *Automorphism commutivity*: for all automorphisms α, β of \mathcal{L} , $\alpha * \beta = \beta * \alpha$. \square

CONVENTION: Throughout the rest of this section let A be the set of automorphisms of \mathcal{L} and let ι be the identity automorphism of \mathcal{L} . \square

The conditions for a scalar structure allow a lot of interaction between the relations of \mathcal{L} and the set of automorphisms A , as is shown in the next definition and three lemmas:

DEFINITION 2.6. Let \mathcal{L} be a scalar structure. Define the binary relation \succeq on A as follows: for each α, β in A ,

$$\alpha \succeq \beta \quad \text{iff} \quad \text{for some } x \text{ in } X, \alpha(x) \succeq \beta(x). \quad \square$$

LEMMA 2.1. Suppose \mathcal{L} is a scalar structure. Then for each α, β in A ,

$$\alpha \succeq \beta \quad \text{iff} \quad \text{for all } y \text{ in } X, \alpha(y) \succeq \beta(y).$$

Proof. Suppose for all y in X , $\alpha(y) \succeq \beta(y)$. Then by Definition 2.6., $\alpha \succeq \beta$.

Suppose $\alpha \succeq \beta$. By Definition 2.6. let x in X be such that $\alpha(x) \succeq \beta(x)$. Let y be an arbitrary element of X . By homogeneity, let γ in A be such that $\gamma(y) = x$. Then $\alpha[\gamma(y)] \succeq \beta[\gamma(y)]$, which by automorphism commutivity yields $\gamma[\alpha(y)] \succeq \gamma[\beta(y)]$, and thus $\alpha(y) \succeq \beta(y)$. \square

LEMMA 2.2. Suppose \mathcal{L} is a scalar structure. Then \succeq is a total ordering on A .

Proof. Let x be an arbitrary element of X and α, β, γ be arbitrary elements of A .

To show \succeq is transitive, suppose $\alpha \succeq \beta$ and $\beta \succeq \gamma$. Then $\alpha(x) \succeq \beta(x)$ and $\beta(x) \succeq \gamma(x)$, and since \succeq is a total ordering on X , it follows that $\alpha(x) \succeq \gamma(x)$. Thus $\alpha \succeq \gamma$.

\succeq is connected since either $\alpha(x) \succeq \beta(x)$ or $\beta(x) \succeq \alpha(x)$ and thus either $\alpha \succeq \beta$ or $\beta \succeq \alpha$.

Now suppose $\alpha \succeq \beta$ and $\beta \succeq \alpha$. Then by Lemma 2.1., for all y in X , $\alpha(y) \succeq \beta(y)$ and $\beta(y) \succeq \alpha(y)$, which since \succeq is a total ordering on X yields $\alpha(y) = \beta(y)$. Thus $\alpha = \beta$. \square

LEMMA 2.3. *Suppose \mathcal{L} is a scalar structure. Then $\mathcal{G} = \langle A, \succeq, * \rangle$ is a (totally) ordered abelian group.*

Proof. It is well-known and easily established that $\langle A, * \rangle$ is a group. It is commutative by assumption.

Let α, β, γ be arbitrary elements of A and let x be an element of X . Then by Lemma 2.1.,

$$\begin{aligned} \alpha \succeq \beta & \text{ iff } \alpha(x) \succeq \beta(x) \\ & \text{ iff } \gamma[\alpha(x)] \succeq \gamma[\beta(x)] \\ & \text{ iff } \gamma * \alpha \succeq \gamma * \beta. \quad \square \end{aligned}$$

CONVENTION. Throughout the rest of this section, when \mathcal{L} is a scalar structure let $\mathcal{G} = \langle A, \succeq, * \rangle$ be the (totally) ordered group of automorphisms of \mathcal{L} . \square

DEFINITION 2.7. Let \mathcal{L} be a scalar structure. Then α in A is said to be *positive* if and only if $\alpha \succ \iota$. \square

DEFINITION 2.8. Let \mathcal{L} be a scalar structure. Then \mathcal{G} is said to be *Archimedean* if and only if for each positive α, β in A , there exists n in I^+ such that $\alpha^n \succ \beta$.

Note that it follows from Lemma 2.1. that the following two statements are equivalent:

- (i) \mathcal{G} is Archimedean;
- (ii) for each α, β in A , if for some x in X , $\alpha(x) \succ x$, then for some n in I^+ , $\alpha^n(x) \succ \beta(x)$. \square

THEOREM 2.4. *Suppose \mathcal{L} is a Dedekind complete scalar structure. Then \mathcal{G} is Archimedean.*

Proof. Suppose \mathcal{G} is not Archimedean. A contradiction will be shown. Let α, β be positive elements of A such that for all n in I^+ , $\beta \succeq \alpha^n$. Let x be an element of X . Then $\beta(x) \succeq \alpha^n(x)$ for all n in I^+ . Thus by Dedekind completeness of \mathcal{L} , let

$$y = \text{l.u.b} \{ \alpha^n(x) \mid n \in I^+ \}.$$

Let m in I^+ be such that

$$y \succ \alpha^{m+1}(x) \succ \alpha^m(x) \succ \alpha^{-1}(y).$$

Then it immediately follows that

$$y \succ \alpha^{m+1}(x) \succ \alpha[\alpha^{-1}(y)] = y,$$

which is impossible. \square

Since features of the qualitative relations of \mathcal{L} (and not features of the automorphisms of \mathcal{L}) are what are usually empirically observed, for measurement-theoretic purposes it is quite desirable to try to express important automorphism properties of \mathcal{L} as directly as possible in terms of the relations of \mathcal{L} . Theorem 2.4. does this somewhat by showing the Archimedeaness of \mathcal{G} is derivable from the Dedekind completeness of \mathcal{L} . A better result from a measurement theoretic point of view would be to give an 'Archimedean condition' in terms of the relations of \mathcal{L} that would be equivalent to \mathcal{G} being Archimedean. This approach will be pursued later in this section.

A simple but important result that is closely related to Theorem 2.4. is given in the following theorem:

THEOREM 2.5. *Suppose \mathcal{L} is a Dedekind complete scalar structure. Then \mathcal{G} is Dedekind complete.*

Proof. Let S be a nonempty subset of A and β in A be such that $\beta \succeq \alpha$ for each α in S . Let x be an element of X . Then $\beta(x) \succeq \alpha(x)$ for each α in S . By the Dedekind completeness of \mathcal{L} , let $y = \text{l.u.b} \{ \alpha(x) \mid \alpha \in S \}$. By homogeneity, let γ in A be such that $\gamma(x) = y$. Let δ be an arbitrary element

of A that is an upper bound of S , i.e., $\delta \in A$ and $\delta \succeq \alpha$ for all α in S . Then for all α in S ,

$$\delta(x) \succeq \gamma = \gamma(x) \succeq \alpha(x),$$

from which it follows that $\delta \succeq \gamma$. Since δ is an arbitrary upper bound of S , it follows that γ is the l.u.b of S . \square

The next main result will be that Dedekind complete scalar structures are ratio scalable. The proof is complex and is based on a method of Cohen and Narens (1979) for constructing an isomorphism between \mathcal{S} and a structure based upon the automorphisms of \mathcal{S} .

DEFINITION 2.9. Define the function \hat{F} and the relation \hat{R} on A as follows: for each α, β, γ in A , $\hat{F}(\alpha, \beta)$ is the function from X into X defined by

$$\hat{F}(\alpha, \beta)(x) = F[\alpha(x), \beta(x)] \text{ for all } x \text{ in } X,$$

and \hat{R} is defined by

$$\hat{R}(\alpha, \beta, \gamma) \quad \text{iff} \quad \text{for all } x \text{ in } X, R[\alpha(x), \beta(x), \gamma(x)]. \quad \square$$

LEMMA 2.4. Suppose \mathcal{S} is a scalar structure and α, β are arbitrary elements of A . Then $\hat{F}(\alpha, \beta)$ is an element of A .

Proof. Let a be a fixed element of X . By homogeneity in \mathcal{S} , let γ be an element of A such that

$$\gamma(a) = F[\alpha(a), \beta(a)] = \hat{F}(\alpha, \beta)(a).$$

It will be shown that $\gamma = \hat{F}(\alpha, \beta)$, and hence that $\hat{F}(\alpha, \beta)$ is in A . Let x be an arbitrary element of X . By homogeneity in \mathcal{S} , let θ in A be such that $\theta(a) = x$. Then by commutativity of automorphisms of A ,

$$\begin{aligned} \gamma(x) &= \gamma[\theta(a)] \\ &= \theta[\gamma(a)] \\ &= \theta(F[\alpha(a), \beta(a)]) \end{aligned}$$

$$\begin{aligned}
&= F(\theta[\alpha(a)], \theta[\beta(a)]) \\
&= F(\alpha[\theta(a)], \beta[\theta(a)]) \\
&= F(\alpha(x), \beta(x)) \\
&= \hat{F}(\alpha, \beta)(x). \quad \square
\end{aligned}$$

LEMMA 2.5. Suppose \mathcal{X} is a scalar structure and α, β, γ are arbitrary elements of A . Then

$$\hat{R}(\alpha, \beta, \gamma) \quad \text{iff} \quad \text{for some } a \text{ in } X, R[\alpha(a), \beta(a), \gamma(a)].$$

Proof. Suppose $\hat{R}(\alpha, \beta, \gamma)$. Then by Definition 2.9., $R[\alpha(a), \beta(a), \gamma(a)]$ for some a in X .

Suppose a in X is such that $R[\alpha(a), \beta(a), \gamma(a)]$. Let x be an arbitrary element of X . By homogeneity in \mathcal{X} , let θ in A be such that $\theta(a) = x$. Then

$$\begin{aligned}
R[\alpha(a), \beta(a), \gamma(a)] &\quad \text{iff} \quad R(\theta[\alpha(a)], \theta[\beta(a)], \theta[\gamma(a)]) \\
&\quad \text{iff} \quad R(\alpha[\theta(a)], \beta[\theta(a)], \gamma[\theta(a)]) \\
&\quad \text{iff} \quad R[\alpha(x), \beta(x), \gamma(x)],
\end{aligned}$$

and thus since x is an arbitrary element of X , $\hat{R}(\alpha, \beta, \gamma)$ holds. \square

LEMMA 2.6. Suppose \mathcal{X} is a scalar structure, a is an element of X , and f is the function from A into X such that for each α in A , $f(\alpha) = \alpha(a)$. Then f is an isomorphism from $\langle A, \succeq, \hat{F}, \hat{R} \rangle$ onto $\langle X, \succeq, F, R \rangle$.

Proof. It immediately follows from the definition of f and Lemma 2.1. that for each α, β in A ,

$$(2.3) \quad \alpha \succeq \beta \quad \text{iff} \quad f(\alpha) \succeq f(\beta).$$

Let α, β, γ be arbitrary elements of A and suppose $\hat{F}(\alpha, \beta) = \gamma$. Then by Definition 2.9., $F[\alpha(a), \beta(a)] = \gamma(a)$, or in other words,

$$(2.4) \quad f(\gamma) = F[f(\alpha), f(\beta)].$$

Let α, β, γ be arbitrary elements of A . Then by Lemma 2.5.,

$$(2.5) \quad \hat{R}(\alpha, \beta, \gamma) \quad \text{iff} \quad R[\alpha(a), \beta(a), \gamma(a)] \\ \text{iff} \quad R[f(\alpha), f(\beta), f(\gamma)].$$

It follows from Equation (2.3) that f is one-to-one. To show f is also onto X , let x be an arbitrary element of X . By homogeneity in \mathcal{L} , let θ be an element of A such that $\theta(a) = x$. Then, $f(\theta) = x$. Thus by Equations (2.3), (2.4), and (2.5), f is an isomorphism. \square

LEMMA 2.7. *Suppose \mathcal{L} is a scalar structure. Then for each $\alpha, \beta, \gamma, \delta$ in A ,*

$$\alpha * \hat{F}(\beta, \gamma) = \hat{F}(\alpha * \beta, \alpha * \gamma)$$

and

$$\hat{R}(\beta, \gamma, \delta) \quad \text{iff} \quad \hat{R}(\alpha * \beta, \alpha * \gamma, \alpha * \delta).$$

Proof. Let $\alpha, \beta, \gamma, \delta$ be arbitrary elements of A . Then for each x in X ,

$$\begin{aligned} [\alpha * \hat{F}(\beta, \gamma)](x) &= \alpha[\hat{F}(\beta, \gamma)(x)] \\ &= \alpha(F[\beta(x), \gamma(x)]) \\ &= F[\alpha * \beta(x), \alpha * \gamma(x)] \\ &= \hat{F}(\alpha * \beta, \alpha * \gamma)(x), \end{aligned}$$

and thus $\alpha * \hat{F}(\beta, \gamma) = \hat{F}(\alpha * \beta, \alpha * \gamma)$. Also,

$$\begin{aligned} \hat{R}(\beta, \gamma, \delta) \quad \text{iff} \quad &\text{for all } x \text{ in } X, R[\beta(x), \gamma(x), \delta(x)] \\ \text{iff} \quad &\text{for all } x \text{ in } X, R[\alpha * \beta(x), \alpha * \gamma(x), \alpha * \delta(x)] \\ \text{iff} \quad &\hat{R}(\alpha * \beta, \alpha * \gamma, \alpha * \delta). \quad \square \end{aligned}$$

THEOREM 2.6. *Suppose \mathcal{L} is a Dedekind complete scalar structure. Then there exists a numerical structure \mathcal{N} such that the following five statements are true:*

1. \mathcal{N} is isomorphic to \mathcal{L} ;
2. the set of automorphisms of \mathcal{N} is the set of all multiplications by positive reals;
3. there exists a \mathcal{N} -representation for \mathcal{L} ;

4. if ϕ is an arbitrary \mathcal{N} -representation for \mathcal{X} and r is an arbitrary positive real, then $r\phi$ is a \mathcal{N} -representation for \mathcal{X} ;
5. if ϕ and ψ are arbitrary \mathcal{N} -representations for \mathcal{X} that are onto \mathcal{N} , then for some s in Re^+ , $s\phi = \psi$.

Proof. It follows from Lemma 2.6. that $\langle A, \succeq \rangle$ is Dedekind complete and dense. It is well-known that all Dedekind complete, densely ordered groups are isomorphic to the ordered multiplicative group of positive reals, $\langle Re^+, \geq, \cdot \rangle$. Let g be an isomorphism from \mathcal{G} onto $\langle Re^+, \geq, \cdot \rangle$. Let $F' = g(\hat{F})$, $R' = g(\hat{R})$, and $\mathcal{N} = \langle Re^+, \geq, \cdot, F', R' \rangle$. Then $\langle A, \succeq, *, \hat{F}, \hat{G} \rangle$ and \mathcal{N} are isomorphic by g .

1. By Lemma 2.6. and the above construction, \mathcal{N} is isomorphic to \mathcal{X} .

2. We will first show that all multiplications by positive reals are automorphisms of \mathcal{N} . Let r be an arbitrary element of Re^+ and let g be the isomorphism of the above construction. Then by using Lemma 2.7. it follows that for each r_1, r_2, r_3 in Re^+ ,

$$\begin{aligned}
 r \cdot F'(r_1, r_2, r_3) &= g[g^{-1}(r)] \cdot g[g^{-1}[F'(r_1, r_2, r_3)]] \\
 &= g[g^{-1}(r) * g^{-1}[F'(r_1, r_2, r_3)]] \\
 &= g[g^{-1}(r) * \hat{F}[g^{-1}(r_1), g^{-1}(r_2), g^{-1}(r_3)]] \\
 &= g[\hat{F}[g^{-1}(r) * g^{-1}(r_1), g^{-1}(r) * g^{-1}(r_2), \\
 &\quad g^{-1}(r) * g^{-1}(r_3)]] \\
 &= F'(g[g^{-1}(r) * g^{-1}(r_1)], g[g^{-1}(r) * g^{-1}(r_2)], \\
 &\quad g[g^{-1}(r) * g^{-1}(r_3)]) \\
 &= F'(r \cdot r_1, r \cdot r_2, r \cdot r_3),
 \end{aligned}$$

and

$$\begin{aligned}
 R'(r_1, r_2, r_3) &\text{ iff } \hat{R}(g^{-1}(r_1), g^{-1}(r_2), g^{-1}(r_3)) \\
 &\text{ iff } \hat{R}[g^{-1}(r) * g^{-1}(r_1), g^{-1}(r) * g^{-1}(r_2), \\
 &\quad g^{-1}(r) * g^{-1}(r_3)] \\
 &\text{ iff } R'(g[g^{-1}(r) * g^{-1}(r_1)], g[g^{-1}(r) * g^{-1}(r_2)], \\
 &\quad g[g^{-1}(r) * g^{-1}(r_3)]) \\
 &\text{ iff } R'(r \cdot r_1, r \cdot r_2, r \cdot r_3),
 \end{aligned}$$

and thus multiplication by r is an automorphism of \mathcal{N} since it also preserves the ordering \geq . To show all automorphisms of \mathcal{N} are multiplications by positive reals, let α be an arbitrary automorphism of \mathcal{N} and $s = \alpha(1)$. By the above, let α_s denote the automorphism of \mathcal{N} that is multiplication by s . Then $\alpha(1) = \alpha_s(1)$, and thus by Lemma 2.1., $\alpha = \alpha_s$.

3. The isomorphism established in Part 1 is a \mathcal{N} -representation for \mathcal{L} .

4. Suppose ϕ is a \mathcal{N} -representation for \mathcal{L} and r is in Re^+ . Then by Part 2, for each x, y, z in X ,

$$\begin{aligned} x \succeq y &\text{ iff } \phi(x) \geq \phi(y) &\text{ iff } r\phi(x) \geq r\phi(y), \\ F(x, y) = z &\text{ iff } F'[\phi(x), \phi(y)] = \phi(z) \\ &\text{ iff } rF'[\phi(x), \phi(y)] = r\phi(z) \\ &\text{ iff } F'[r\phi(x), r\phi(y)] = r\phi(z), \end{aligned}$$

and

$$\begin{aligned} R(x, y, z) &\text{ iff } R'[\phi(x), \phi(y), \phi(z)] \\ &\text{ iff } R'[r\phi(x), r\phi(y), r\phi(z)], \end{aligned}$$

from which it follows that $r\phi$ is a \mathcal{N} -representation.

5. Suppose ϕ and ψ are \mathcal{N} -representations for \mathcal{L} that are onto \mathcal{N} . Since ϕ and ψ are one-to-one and onto Re^+ , $\phi\psi^{-1}$ is an automorphism of \mathcal{N} , and thus by Part 2, let r in Re^+ be such that $\phi\psi^{-1}$ is multiplication by r . Then for each x in X ,

$$\phi(x) = \phi\psi^{-1}[\psi(x)] = r\psi(x). \quad \square$$

Let \mathcal{L} be a Dedekind complete scalar structure. Theorem 2.6. establishes the existence of what is commonly called a 'ratio scaling' of \mathcal{L} . In our setup, this 'ratio scaling' is precisely defined as an ordered pair $\langle \eta, \mathcal{N} \rangle$ where η is a \mathcal{N} -representation of \mathcal{L} and \mathcal{N} satisfies Statements 1 to 5 of Theorem 2.6. Thus two ratio scalings $\langle \eta, \mathcal{N} \rangle$ and $\langle \eta_1, \mathcal{N}_1 \rangle$ of \mathcal{L} can differ either in terms of their numerical imbeddings ($\eta \neq \eta_1$) or in terms of their representing structures ($\mathcal{N} \neq \mathcal{N}_1$).

Suppose \mathcal{L} is a Dedekind complete scalar structure and $\mathcal{N} = \langle Re^+, \geq, F', R' \rangle$ and $\mathcal{N}_1 = \langle Re^+, \geq, F_1, R_1 \rangle$ satisfy Statements 1 and 2 of Theorem 2.6. Consider $\mathcal{M} = \langle Re^+, \geq, F', R', \cdot \rangle$ and $\mathcal{M}_1 = \langle Re^+, \geq, F_1, R_1, \cdot \rangle$. Then both \mathcal{M} and \mathcal{M}_1 are isomorphic to $\langle A, \succeq, \hat{F}, \hat{R}, * \rangle$, and thus \mathcal{M} and \mathcal{M}_1 are

themselves isomorphic. Let g be an isomorphism of \mathcal{M} onto \mathcal{M}_1 . Then g is an automorphism of $\langle Re^+, \geq, \cdot \rangle$, and thus we can let r in Re^+ be such that $g(w) = w^r$ for each w in Re^+ . Then for each t, u, v in Re^+ ,

$$F'(t, u) = v \quad \text{iff} \quad F_1(t^r, u^r) = v^r,$$

and

$$R'(t, u, v) \quad \text{iff} \quad R_1(t^r, u^r, v^r).$$

Suppose ϕ and ψ are respectively \mathcal{N} and \mathcal{N}_1 representations for \mathcal{X} that are onto Re^+ . Then for each x, y, z in X ,

$$\begin{aligned} F(x, y) = z & \quad \text{iff} \quad F'[\phi(x), \phi(y)] = \phi(z) \\ & \quad \text{iff} \quad F_1[\phi(x)^r, \phi(y)^r] = \phi(z)^r. \end{aligned}$$

$$\begin{aligned} R(x, y, z) & \quad \text{iff} \quad R'[\phi(x), \phi(y), \phi(z)] \\ & \quad \text{iff} \quad R_1[\phi(x)^r, \phi(y)^r, \phi(z)^r], \end{aligned}$$

and

$$x \succeq y \quad \text{iff} \quad \phi(x) \geq \phi(y) \quad \text{iff} \quad \phi(x)^r \geq \phi(y)^r.$$

Thus ϕ^r is a \mathcal{N}_1 -representation for X and, ϕ^r is onto \mathcal{N}_1 . Therefore by Statement 5 of Theorem 2.6 (which is a consequence of Statements 1 and 2), for some s in Re^+ , $\psi = s\phi^r$.

Suppose \mathcal{X} is a Dedekind complete scalar structure and $\mathcal{N} = \langle Re^+, \geq, F', R' \rangle$ satisfies Statements 1 and 2 of Theorem 2.6. Let u, v be arbitrary elements of Re^+ and $\eta = u\phi^v$. Then $\eta(X) = Re^+$. Let $F_2 = \eta(F)$, $R_2 = \eta(R)$, and $\mathcal{M} = \langle Re^+, \geq, F_2, R_2 \rangle$. Then η is a \mathcal{M} -representation for X and η is onto \mathcal{M} . Let h be the function from Re^+ onto Re^+ such that $h(t) = ut^v$ for all t in Re^+ . Then each automorphism of \mathcal{M} is of the form $h(\alpha)$ where α is some automorphism of \mathcal{N} . Let $\alpha_p(t) = pt, p > 0$, be an arbitrary automorphism of \mathcal{N} . Let $\beta = h(\alpha_p)$. Then for each t in Re^+ ,

$$\beta[h(t)] = h[\alpha_p(t)],$$

and thus

$$\beta(ut^v) = u(pt)^v,$$

and letting $q = p^v$ and $w = ut^v$, this latter equation yields

$$\beta(w) = qw$$

for all w in Re^+ . Thus \mathcal{M} has multiplications by the positive reals as its automorphisms.

The above results are summarized in the following theorem:

THEOREM 2.7. *Suppose \mathcal{L} is a Dedekind complete scalar structure, $\mathcal{N} = \langle Re^+, \geq, F', R' \rangle$, $\mathcal{N}_1 = \langle Re^+, \geq, F_1, R_1 \rangle$, and ϕ and ψ are respectively \mathcal{N} and \mathcal{N}_1 -representations for \mathcal{L} that are onto Re^+ and have their automorphisms coinciding with multiplications by the positive reals. Then there exists r, s in Re^+ such that the following four statements hold for all x in X and all u, v, w in Re^+ :*

1. $F'(u, v) = F_1(u^r, v^r)^{1/r}$;
2. $R'(u, v, w)$ iff $R_1(u^r, v^r, w^r)$;
3. $\psi(x) = s\phi(x)^r$;
4. *there exists \mathcal{M} such that $u\phi^v$ is a \mathcal{M} -representation for \mathcal{L} that is onto \mathcal{M} and has multiplications by the positive reals as its automorphisms.*

Proof. Statements 1, 2, 3, and 4 follow the above discussion.

Often in science, phenomena are assumed to be ratio scalable without giving consideration to the qualitative assumption underlying such scalings. The commonly used method is to assume that a ratio scaling ϕ exists and that any other ratio scaling ψ is of the form $s\phi$ for some positive real s . However, from Theorem 2.7 it seems to follow that 'any ratio scaling ψ ' should be of the form $s\phi^r$ which, by taking logarithms, yields a family of representations of the form $r \log(\phi) + \log s$, which is the familiar form of interval scalability. In this case, the automorphisms of the qualitative (scalar) structure have quite different properties than those of the family of representations. To my knowledge, no one has investigated the measurement-theoretic properties of this more general form of 'ratio scalability'.

In the above development of scalar structures, the critical use of the Dedekind completeness of \mathcal{L} was in the establishment of the isomorphism of $\langle A, \succeq, * \rangle$ onto $\langle Re^+, \geq, \cdot \rangle$. This was done by first showing that $\langle A, \succeq \rangle$ is Dedekind complete (Theorem 2.5.), and then using the well-known

theorem that all Dedekind complete, totally ordered groups are isomorphic to $\langle Re^+, \geq, \cdot \rangle$. A weaker assumption to the Dedekind completeness of \mathcal{L} is that \mathcal{G} is Archimedean (Theorem 2.4.). If instead of the Dedekind completeness of \mathcal{L} , the Archimedeaness of \mathcal{G} is assumed, then by another well-known theorem – Hölder’s Theorem – there exists an isomorphic imbedding of $\langle A, \succeq, * \rangle$ into $\langle Re^+, \geq, \cdot \rangle$. Using this result and making some minor modifications in the proofs, theorems analogous to Theorems 2.6. and 2.7. go through with the Archimedeaness of \mathcal{G} replacing the hypothesis of the Dedekind completeness of \mathcal{L} . (In the statement of Theorem 2.6., Statement 2 should read “The set of automorphisms of \mathcal{N} is a subset of the set of all multiplications by positive reals”, and Statement 4 should be deleted. In the statement of Theorem 2.7., the domains of discourse for \mathcal{N} and \mathcal{N}_1 should be changed from Re^+ to N and N_1 respectively, where N and N_1 are subsets of Re^+ , and the range of the variables u, v , and w should be changed from Re^+ to N , and Statement 4 should be deleted.)

We will now give sufficient conditions on \mathcal{L} that allow it to extend nicely to a Dedekind complete scalar structure. These conditions will identify F and R (so that \mathcal{L} is essentially a structure of the form $\langle X, \succeq, F \rangle$) and require that F be strictly increasing in each variable. Because of the additional structure this latter condition brings, some of the automorphism conditions for a scalar structure can be weakened.

DEFINITION 2.10. \mathcal{L} is said to be a *monotonic prescalar structure* if and only if the following conditions hold for each x, y, z in \mathcal{L} :

- (i) $F(x, y) = z$ iff $R(x, y, z)$;
- (ii) *automorphism density*: if $x \succ y$, then for some α in A , $x \succ \alpha(z) \succ y$;
- (iii) *automorphism ordering*: for each α in A , if $\alpha(u) \succ u$ for some u in X , then $\alpha(v) \succ v$ for all v in X ;
- (iv) *monotonicity*: $x \succeq y$ iff $F(x, z) \succeq F(y, z)$
iff $F(z, x) \succeq F(z, y)$;
- (v) *A-Archimedean*: for each α in A , if $\alpha(x) \succ x$, then for some n in I^+ , $\alpha^n(x) \succ y$. \square

Let X be a monotonic prescalar structure. By automorphism ordering, a relation \succeq can be defined on A in a manner analogous to Definition 2.6., and can be shown to be a total ordering in a manner similar to Lemma 2.2. It also easily follows that $\mathcal{G} = \langle A, \succeq, * \rangle$ is a totally ordered group. Using A -Archimedean, it then follows that \mathcal{G} is Archimedean. It is a consequence of Hölder's theorem that all Archimedean, totally ordered groups are commutative. Thus the automorphisms of \mathcal{A} commute. It follows from automorphism density that \succeq is a dense ordering on X . Thus \mathcal{A} looks very much like a scalar structure except that it satisfies the condition automorphism density, which is analogous to but weaker than homogeneity. The following example shows that monotonic prescalar structures need not be scalar structures:

EXAMPLE 2.1. Let Ra^+ be the positive rationals, π a positive transcendental real number, and

$$X = \{x\pi + y \mid x \in Ra^+ \cup \{0\} \text{ and } y \in Ra^+\}.$$

For each x, y, z in X , let $F(x, y) = z$ and $R(x, y, z)$ if and only if $x + y = z$. Then it can be shown that $\mathcal{A} = \langle X, \succcurlyeq, F, R \rangle$ is a prescalar structure. \mathcal{A} is not a scalar structure since it is not homogeneous: there is no automorphism α of \mathcal{A} such that $\alpha(1) = \pi$. \square

THEOREM 2.8. Suppose \mathcal{A} is a monotonic prescalar structure. Then \mathcal{A} has an extension $\bar{\mathcal{A}} = \langle \bar{X}, \succeq', \bar{F}, \bar{R} \rangle$ that has the following five properties:

- (i) $\bar{\mathcal{A}}$ is Dedekind complete;
- (ii) $\bar{\mathcal{A}}$ is homogeneous, i.e., for each x, y in X there exists an automorphism α of $\bar{\mathcal{A}}$ such that $\alpha(x) = y$;
- (iii) the extension \bar{F} satisfies monotonicity (Definition 2.10.);
- (iv) each automorphism of \mathcal{A} extends to an automorphism of $\bar{\mathcal{A}}$;
- (v) X is dense in $\langle \bar{X}, \succeq' \rangle$, i.e., for each x, y in \bar{X} , if $x \succ' y$, then there exists z in X such that $x \succ' z \succ' y$.

Proof. See Theorem 5.2. \square

Let \mathcal{L} be a scalar structure. In the above development of the ratio scalability of \mathcal{L} , automorphisms of \mathcal{L} play a very important role. Although the concept ‘automorphism of \mathcal{L} ’ cannot be defined in an appropriate first order language for \mathcal{L} , this does not mean that individual automorphisms are not expressible in such a language: the identity automorphism, ι , is expressible in such a language, and if α and β are first order expressible then so is the automorphism $\hat{F}(\alpha, \beta)$ (see Definition 2.9. and Lemma 2.4.). Except for degenerate cases, $\hat{F}(\alpha, \beta)$ will be different from α and β , and it is usually the case that infinitely many automorphisms of \mathcal{L} are definable in this manner. This method is very general, for if $D(x_1, \dots, x_n)$ is any first order definable (in terms of \mathcal{L}) n -ary function from X into X , then analogously to Definition 2.9. and Lemma 2.4., \hat{D} can be defined and $\hat{D}(\alpha_1, \dots, \alpha_n)$ can be shown to be an automorphism of \mathcal{L} for all $\alpha_1, \dots, \alpha_n$ in A . The existence of lots of definable automorphisms make it feasible to axiomatize certain scalar and monotonic prescalar structures in a manner in which automorphisms are not directly mentioned.

Write $x \circ y$ for $F(x, y)$ and suppose that $F(x, y) = z$ iff $R(x, y, z)$ for all x, y, z in X . Then \mathcal{L} is essentially the structure $\langle X, \succeq, \circ \rangle$. Suppose \mathcal{L} satisfies the axioms for a positive concatenation structure (Definition 1.4.). From results of Cohen and Narens (1979) it follows that \mathcal{L} satisfies all the conditions for a monotonic prescalar structure except possibly for automorphism density. However, it also follows from results of Cohen and Narens (1979) that if

$$n(x \circ y) = (nx) \circ (ny)$$

for each n in I^+ and each x in X – a condition that is expressible by an infinite set of first order axioms – then automorphism density is also satisfied.

Cohen and Narens (1979) also showed that positive concatenation structures \mathcal{L} that satisfies automorphism density are extendable to positive concatenation structures that are Dedekind complete scalar structures. The methods developed in their paper can be extended to show the following two theorems:

THEOREM 2.9. *Let $\mathcal{L} = \langle X, \succeq, F, R \rangle = \langle X, \succeq, \circ \rangle$ be a positive concatenation structure in the sense described in the previous two paragraphs. Then the following three conditions are equivalent:*

- (i) \mathcal{L} satisfies automorphism density (Definition 2.10.);
- (ii) for each x, y in X , if $x \succ y$, then for some α in A , $x \succ \alpha(y) \succ y$;
- (iii) for each n in I^+ and each x, y in X , $n(x \circ y) = (nx) \circ (ny)$.

Proof. See Theorem 5.3. \square

THEOREM 2.10. *Let \mathcal{L} be as in Theorem 2.9. Suppose \mathcal{L} satisfies automorphism density. Then the following three statements are true:*

1. \mathcal{L} is extendable to a positive concatenation structure that is a Dedekind complete scalar structure (Cohen & Narens).
2. All Dedekind completions of \mathcal{L} that are positive concatenation structures are isomorphic.
3. If \mathcal{F} is a Dedekind completion of \mathcal{L} that is a positive concatenation structure, then each automorphism α of \mathcal{L} extends to an automorphism of \mathcal{F} .

Proof. See Theorem 5.4. \square

It easily follows from the axioms for a monotonic prescalar structure that a natural ordering \succeq can be defined on A so that $\mathcal{G} = \langle A, \succeq, * \rangle$ is an Archimedean totally ordered group, and these properties of \mathcal{G} play a critical role in the development of monotonic prescalar structures. As in positive concatenation structures, these assumptions about A in some cases can be replaced by axioms involving the relations of \mathcal{L} ; in other cases – as the next theorem shows – they can be replaced by assumptions concerning numerical representations of \mathcal{L} :

DEFINITION 2.11. A set of automorphisms H of \mathcal{L} is said to satisfy *1-point uniqueness* if and only if for each α, β in H , if $\alpha(x) = \beta(x)$ for some x in X , then $\alpha = \beta$.

\mathcal{L} is said to satisfy *1-point \mathcal{N} -uniqueness* if and only if (i) there exists a \mathcal{N} -representation for \mathcal{L} , and (ii) for all \mathcal{N} -representations ϕ, ψ of \mathcal{L} , if $\phi(x) = \psi(x)$ for some x in X , then $\phi = \psi$. \square

THEOREM 2.11. *The following two propositions are true:*

1. *Suppose \mathcal{L} satisfies 1-point \mathcal{N} -uniqueness. Then A satisfies 1-point uniqueness.*
2. *Suppose A satisfied 1-point uniqueness and $\langle X, \succeq \rangle$ is Dedekind complete and satisfies density (Definition 2.5.). Let \succeq be the binary relation on A that is defined as follows: for each α, β in A ,*

$$\alpha \succeq \beta \quad \text{iff} \quad \text{for some } x \text{ in } X, \alpha(x) \succeq \beta(x).$$

*Then $\mathcal{G} = \langle A, \succeq, * \rangle$ is a totally ordered group that is Archimedean and commutative.*

Proof. 1. Suppose α, β are elements of A , x is in X , and $\alpha(x) = \beta(x)$. By 1-point \mathcal{N} -uniqueness, let ϕ and \mathcal{N} be such that ϕ is a \mathcal{N} -representation for \mathcal{L} . Then $\phi\alpha$ and $\phi\beta$ are also \mathcal{N} -representations of \mathcal{L} and $\phi\alpha(x) = \phi\beta(x)$. Thus by 1-point \mathcal{N} -uniqueness, $\phi\alpha = \phi\beta$. Since ϕ , α , and β are one-to-one, $\alpha = \beta$.

2. We first show the conclusion of Lemma 2.1., that for all y in X , $\alpha \succeq \beta$ iff $\alpha(y) \succeq \beta(y)$. If $\alpha(y) \succeq \beta(y)$ for all y in X , then it is immediate that $\alpha \succeq \beta$. Suppose $\alpha \succeq \beta$. Let x in X be such that $\alpha(x) \succeq \beta(x)$. Let y be an arbitrary element of X . To show Proposition 2, we need only show that $\alpha(y) \prec \beta(y)$ leads to a contradiction. Suppose $\alpha(y) \prec \beta(y)$. Then $x \neq y$. Let $\tau = \beta^{-1}\alpha$. Then $\tau(y) \prec y$ and $\tau(x) \succeq x$. Since $\langle X, \succeq \rangle$ is dense and Dedekind complete, it is isomorphic to $\langle Re^+, \succeq \rangle$, and since τ is one-to-one, onto, and order preserving, the image of τ under this isomorphism is a continuous function on Re^+ . Thus by isomorphism and the intermediate value theorem of calculus, let z in X be such that $\tau(z) = z$. Since the identity, ι , is in A and $\iota(z) = z$, by 1-point uniqueness $\tau = \iota$, which contradicts $\tau(y) \prec y$.

To show that \succeq is a total ordering on A , we use the proof of Lemma 2.2., which requires only the conclusion of Lemma 2.1. and that \succeq is a total ordering on X .

To show that $\mathcal{G} = \langle A, \succeq, * \rangle$ is Archimedean, observe that either $A = \{\iota\}$ or $A \neq \{\iota\}$. In the first case, A is immediately Archimedean, and in the second we use the proof of Theorem 2.4.

\mathcal{G} is commutative since it is well-known that all Archimedean totally ordered groups are commutative. \square

Much of the development of scalar structures can be generalized to the case where a subgroup H of automorphisms of \mathcal{X} (rather than the group A of automorphisms of \mathcal{X}) satisfy homogeneity and automorphism commutativity (or their equivalents). The resulting theorem about ratio scalability for this generalization yields representations onto a numerical structure $\mathcal{N} = \langle \text{Re}^+, \geq, F', R' \rangle$ where automorphisms in H correspond to automorphisms of \mathcal{N} that are multiplications by positive reals. The next definition and theorem gives an explicit statement of this result.

DEFINITION 2.12. H is said to be a *scalar subgroup* of A if and only if H is a subgroup of A that satisfies the following two conditions:

- (i) *H-homogeneity*: for each x, y in X there exists α in H such that $\alpha(x) = y$;
- (ii) *1-point uniqueness*: for each α, β in H if $\alpha(x) = \beta(x)$ for some x , then $\alpha = \beta$. \square

THEOREM 2.12. Suppose $\langle \mathcal{X}, \succeq \rangle$ is Dedekind complete and satisfies density (Definition 2.5.) and H is a scalar subgroup of A . Then there exists a numerical structure $\mathcal{N} = \langle \text{Re}^+, \geq, F', R' \rangle$ and a function ϕ from X onto Re^+ such that the following three statements are true:

1. ϕ is a \mathcal{N} -representation for \mathcal{X} ;
2. for each α in H , $\phi(\alpha)$ is an automorphism of \mathcal{N} that is multiplication by a positive real; and
3. each multiplication by a positive real is an automorphism β of \mathcal{N} such that $\phi^{-1}(\beta)$ is in H .

Proof. Define \succeq on H as follows: for each α, β in H , $\alpha \succeq \beta$ if and only if for some x in X , $\alpha(x) \geq \beta(x)$. By arguments very similar to the proof of Theorems 2.11. and 2.5., it follows that $\langle H, \succeq, * \rangle$ is a Dedekind complete, totally ordered group. (Since all Dedekind complete, totally ordered groups are commutative, H is also commutative.) Statements 1, 2 and 3 then follow by modifying the proof of Statements 1 and 2 of Theorem 2.6. by replacing in that proof 'A' by 'H' and 'automorphism of X ' by 'element of H ' and 'automorphism of \mathcal{N} ' by 'element of $\phi(H)$ for the \mathcal{N} -representation ϕ of X '. \square

It should be noted that the results about \mathcal{R} up to Definition 2.10. do not depend in any essential way on the nature of F and R , and will hold for any other pair of n -ary and m -ary functions and relations. The proof of Theorem 2.8. about Dedekind complete extensions of monotonic prescalar \mathcal{R} is quite general and easily extends to monotonic prescalar-like structures with any number of functions that are strictly increasing in each argument.

3. QUALITATIVE MEANINGFULNESS

Qualitative meaningfulness is a term used to describe relations and concepts that are relevant to the underlying measurement situation. 'Relevant' here is used loosely since no one has really described what properties a satisfactory meaningfulness concept should have. What has happened is that various researchers have invoked various meaningfulness concepts for particular measurement contexts. In this paper, I will take 'meaningfulness' to refer to those properties and concepts that are consistent with the particular underlying measurement situation, where by 'consistent' I mean that the relevant qualitative properties and concepts are incorporatable into the qualitative structure without changing the underlying measurement situation. For example, one might consider the relation T on Y to be meaningful for the structure $\langle Y, P, Q \rangle$ if and only if $\langle Y, P, Q \rangle$ and $\langle Y, P, Q, T \rangle$ satisfy the same measurement processes, where 'measurement processes' for one kind of meaningfulness may be taken as the set of \mathcal{N} -representations for a particular \mathcal{N} , and for another kind of meaningfulness as the set of automorphisms of \mathcal{Y} , etc.

CONVENTION: Throughout the rest of this section, let $\mathcal{Y} = \langle Y, R_1, R_2, \dots \rangle$ be a relational structure, where for each i , R_i is a n_i -ary relation on Y . (Note that we do not require R_1 to be an ordering relation on Y .) \square

DEFINITION 3.1. ϕ is said to be a \mathcal{N} -representation for \mathcal{Y} if and only if ϕ is a homomorphism from \mathcal{Y} into \mathcal{N} . \square

In Definition 3.1., \mathcal{N} is not required to be a numerical structure since all the proofs in this section hold in this more general context.

Pfanzagl (1968) has investigated a qualitative form of meaningfulness

that bears a strong resemblance to quantitative meaningfulness as given in Definition 2.4. This qualitative type of meaningfulness is given in the following definition:

DEFINITION 3.2. A n -ary relation S on Y is said to be *qualitatively \mathcal{N} -meaningful* if and only if there exists a n -ary relation T on the domain of discourse, N , of \mathcal{N} such that for all \mathcal{N} -representations ϕ of \mathcal{Y} and all y_1, \dots, y_n in Y ,

$$S(y_1, \dots, y_n) \text{ iff } T[\phi(y_1), \dots, \phi(y_n)]. \quad \square$$

Note that the relation T in Definition 3.2., if it exists, has the property that for all y_1, \dots, y_n in Y and all \mathcal{N} -representations ϕ, ψ of \mathcal{Y} ,

$$T[\phi(y_1), \dots, \phi(y_n)] \text{ iff } T[\psi(y_1), \dots, \psi(y_n)],$$

so that for the case of $\mathcal{Y} = \mathcal{Z} = \langle X, \succeq, F, R \rangle$ and $\mathcal{N} = \langle N, \succcurlyeq, F', R' \rangle$ and $N \subseteq Re^+$, T is quantitatively \mathcal{N} -meaningful for \mathcal{Y} (Definition 2.4.). Thus qualitative \mathcal{N} -meaningfulness is the natural corresponding qualitative concept to quantitative \mathcal{N} -meaningfulness.

Suppose ϕ is a \mathcal{N} -representation of \mathcal{Y} that is one-to-one and onto \mathcal{N} , i.e., ϕ is an isomorphism from \mathcal{Y} onto \mathcal{N} . Then in a natural way all \mathcal{N} -representations of \mathcal{Y} correspond to endomorphisms of \mathcal{Y} and visa versa. Thus in this case, qualitative \mathcal{N} -meaningfulness of relations on Y becomes invariance under endomorphisms of \mathcal{Y} . This suggests taking invariance under endomorphisms as a definition of meaningfulness, and this approach has been taken by Krantz *et al.* (in preparation) and Luce (1979). It should be noted that such an approach is purely qualitative and makes no reference to a representing structure.

DEFINITION 3.3. A n -ary relation S on Y is said to be *endomorphism meaningful* if and only if for each endomorphism γ of \mathcal{Y} and each y_1, \dots, y_n in Y ,

$$S(y_1, \dots, y_n) \text{ iff } S[\gamma(y_1), \dots, \gamma(y_n)]. \quad \square$$

Another purely qualitative approach to meaningfulness is to consider invariance of relations on Y under automorphisms of \mathcal{Y} :

DEFINITION 3.4. A n -ary relation S on Y is said to be *automorphism meaningful* if and only if for each automorphism γ of \mathcal{Y} and each y_1, \dots, y_n in Y ,

$$S(y_1, \dots, y_n) \text{ iff } S[\gamma(y_1), \dots, \gamma(y_n)]. \quad \square$$

The following theorem describes the logical relationships of the above three types of meaningfulness:

THEOREM 3.1. *Let S be a n -ary relation on Y . Then the following two statements are true:*

1. *If for some \mathcal{N} , a \mathcal{N} -representation exists and S is qualitatively \mathcal{N} -meaningful, then S is endomorphism meaningful.*
2. *If S is endomorphism meaningful, then S is automorphism meaningful.*

Proof. 1. Suppose ϕ is a \mathcal{N} -representation for \mathcal{Y} , γ is an arbitrary endomorphism of \mathcal{Y} , and T is a n -ary relation on the domain of discourse of \mathcal{N} such that for all \mathcal{N} -representations ψ of \mathcal{Y} and all y_1, \dots, y_n in Y ,

$$S(y_1, \dots, y_n) \text{ iff } T[\psi(y_1), \dots, \psi(y_n)].$$

But $\phi' = \phi\gamma$ is also a \mathcal{N} -representation for \mathcal{Y} , and thus by the qualitative \mathcal{N} -meaningfulness of S , for each y_1, \dots, y_n in Y ,

$$\begin{aligned} S(y_1, \dots, y_n) &\text{ iff } T[\phi(y_1), \dots, \phi(y_n)] \\ &\text{ iff } T[\phi'(y_1), \dots, \phi'(y_n)] \\ &\text{ iff } T[\phi\gamma(y_1), \dots, \phi\gamma(y_n)] \\ &\text{ iff } S[\gamma(y_1), \dots, \gamma(y_n)]. \end{aligned}$$

Thus since γ is an arbitrary endomorphism of \mathcal{Y} , the endomorphism meaningfulness of S has been shown.

2. Since all automorphisms of \mathcal{Y} are endomorphisms of \mathcal{Y} , it immediately follows that the endomorphism meaningfulness of S implies the automorphism meaningfulness of S . \square

The following two examples show that the converses of statements 1 and 2 of Theorem 3.1. need not hold:

EXAMPLE 3.1. Let $Z = \{1, 2, 3\}$. Then $\mathcal{X} = \langle Z, \geq \rangle$ has only the identity endomorphism, ι , and thus every relation on Z is endomorphism meaningful. In particular, the 1-ary relation S on Z , where for each z in Z , $S(z)$ holds if and only if $z = 2$, is endomorphism meaningful. Let $\mathcal{N} = \langle Re^+, \geq \rangle$. Then the identity map on Z , ι , is a \mathcal{N} -representation for \mathcal{X} . We will show by contradiction that S is not qualitatively \mathcal{N} -meaningful. Suppose S were qualitatively \mathcal{N} -meaningful. Let T be a 1-ary relation on Re^+ such that for all \mathcal{N} -representation ϕ of \mathcal{X} ,

$$S(z) \quad \text{iff} \quad T[\phi(z)]$$

for all z in Z . Let ψ be the following \mathcal{N} -representation for Z : $\psi(1) = 1$, $\psi(2) = 3$, $\psi(3) = 4$. Then since ι is a \mathcal{N} -representation for \mathcal{X} ,

$$(3.1) \quad S(3) \quad \text{iff} \quad T(3).$$

Since ψ is a \mathcal{N} -representation for \mathcal{X} ,

$$(3.2) \quad S(2) \quad \text{iff} \quad T(3).$$

Thus from Expressions (3.1) and (3.2), $S(3)$ iff $S(2)$, which is impossible. \square

The next example shows that automorphism meaningfulness and endomorphism meaningfulness need not coincide – even for positive concatenation structures with dense automorphism groups:

EXAMPLE 3.2. Let π be a positive transcendental real number and $P = \{1, \pi, \pi^2, \pi^3, \dots\}$. Let R be the intersection of all sets Q such that

- (i) $P \subseteq Q$,
- (ii) if r is a positive rational, then $ry \in Q$, and
- (iii) if $y, z \in Q$, then $y + z \in Q$.

Then $\langle R, \geq, + \rangle$ is an extensive structure. For each positive real r , let α_r be the function from R into Re^+ defined by $\alpha_r(y) = ry$. Then, by construction, for each positive rational r , α_r is an automorphism of $\langle R, \geq, + \rangle$. From this it follows that $\langle R, \geq, + \rangle$ has a dense automorphism group. Now α_π is an endomorphism of $\langle R, \geq, + \rangle$ that is not an automorphism of $\langle R, \geq, + \rangle$ since there is no element y of R such that $\alpha_\pi(y) = 1$. Let Ra^+ be the set of positive

rational. Ra^+ will be shown automorphism meaningful by showing all automorphisms of $\langle R, \geq, + \rangle$ are multiplications by positive rationals. Suppose α_r is an automorphism of $\langle R, \geq, + \rangle$. Then $\alpha_r(1) = r \in R$. Thus by the method of construction of R it follows that $r = p(\pi)$ where $p(x)$ is a polynomial in x with positive rational coefficients. Since α_r is an automorphism, α_r^{-1} is also an automorphism, and $\alpha_r^{-1}(1) = 1/r = q(\pi)$, where $q(x)$ is a polynomial in x with positive rational coefficients. However, since $p(\pi)q(\pi) = 1$ and π is transcendental, p and q must be polynomials of degree 0, i.e., r is a positive rational. Since it can be shown using Theorem 1.2. that all automorphisms of $\langle R, \geq, + \rangle$ are of the form α_r for some r , it follows that Ra^+ is automorphism meaningful. Ra^+ is not endomorphism meaningful since $1 \in Ra^+$ and $\alpha_\pi(1) = 1 \notin Ra^+$. \square

THEOREM 3.2. *Suppose \mathcal{Y} and \mathcal{N} satisfy the following three conditions:*

- (i) Existence: *There exists a \mathcal{N} -representation for \mathcal{Y} .*
- (ii) Uniqueness: *For all \mathcal{N} -representations ϕ, ψ of \mathcal{Y} , if for some y in Y , $\phi(y) = \psi(y)$, then $\phi = \psi$.*
- (iii) \mathcal{Y} -homogeneity: *For each y, z in Y there exists an automorphism α of \mathcal{Y} such that $\alpha(y) = z$.*

Then the concepts of automorphism meaningfulness, endomorphism meaningfulness, and \mathcal{N} -qualitative meaningfulness coincide.

Proof. Let S be a n -ary relation on Y . By Theorem 3.1., to show the theorem we need only show the automorphism meaningfulness of S implies the \mathcal{N} -qualitative meaningfulness of S . (Note Theorem 3.1. assumes existence.) Suppose S is automorphism meaningful. For each \mathcal{N} -representation ϕ of \mathcal{Y} , let S_ϕ be the relation on the domain of discourse of \mathcal{N} defined by: for each y_1, \dots, y_n in Y ,

$$(3.3) \quad S_\phi[\phi(y_1), \dots, \phi(y_n)] \quad \text{iff} \quad S(y_1, \dots, y_n).$$

Let $\mathcal{F} = \{\phi \mid \phi \text{ is a } \mathcal{N}\text{-representation for } \mathcal{Y}\}$. By existence \mathcal{F} is non-empty. Let

$$T = \bigcup_{\phi \in \mathcal{F}} S_\phi.$$

Let η be an arbitrary \mathcal{N} -representation of \mathcal{Y} and y_1, \dots, y_n be arbitrary elements of Y . To show S is qualitatively \mathcal{N} -meaningful, we need only show that

$$S(y_1, \dots, y_n) \text{ iff } T[\eta(y_1), \dots, \eta(y_n)].$$

Suppose $S(y_1, \dots, y_n)$. Then by definition of T , $T[\eta(y_1), \dots, \eta(y_n)]$. Now suppose $T[\eta(y_1), \dots, \eta(y_n)]$. Let ϕ be a \mathcal{N} -representation for \mathcal{Y} such that $S_\phi[\eta(y_1), \dots, \eta(y_n)]$. Then for some z_1 in Y , $\eta(y_1) = \phi(z_1)$. By \mathcal{Y} -homogeneity, let α be an automorphism of \mathcal{Y} such that $\alpha(y_1) = z_1$. Then $\phi\alpha$ is a \mathcal{N} -representation for \mathcal{Y} , and $\phi\alpha(y_1) = \eta(y_1)$. By uniqueness, $\phi\alpha = \eta$. Thus $S_\phi[\phi\alpha(y_1), \dots, \phi\alpha(y_n)]$, which by Equation (3.3) yields $S[\alpha(y_1), \dots, \alpha(y_n)]$, which by the automorphism meaningfulness of S yields $S(y_1, \dots, y_n)$. \square

In our development, the qualitative structure \mathcal{Y} is intended to represent some empirical phenomena, and as such its primitive relations and functions represent observable phenomena and are to be thought of as empirical entities. But what of the measurement process, can it be thought of as 'empirical'? Since measurement processes generally involve mathematical entities, they generally have nonempirical components. But as we have seen, many important measurement concepts have corresponding concepts formulated in terms of automorphisms (or endomorphisms) of the qualitative structure, and so it seems reasonable to first approach this problem qualitatively and ask, "Can automorphisms of empirical structures be thought of as empirical processes?" If they can, then they can be added to the empirical structure \mathcal{Y} to produce a closely related empirical structure \mathcal{Y}' . Obviously in this case the measurement process for \mathcal{Y}' will be more restrictive or just as restrictive as for \mathcal{Y} since \mathcal{Y}' has more structure to be preserved than \mathcal{Y} . If we take 'measurement process' and 'meaningfulness' as intuitive concepts, then in line with our development we would expect the following condition to be necessary for the coincidence of the measurement processes for \mathcal{Y} and \mathcal{Y}' : All automorphisms of \mathcal{Y} are meaningful. Let's consider these matters in a more concrete setting.

Consider physical measurement. Here we are dealing with a system ultimately reducible to interactions of certain basic structures called "units" which are ratio scalable. Length is one of the unit structures, and theoretically it can be considered as a closed extensive structure $\mathcal{L} = \langle L, \succeq, \circ \rangle$. Clearly the primitives L, \succeq , and \circ are to be treated as physical, and it can easily be argued that certain concepts that are simply defined in terms of the primitives of \mathcal{L} , e.g., α_2 where $\alpha_2(y) = y \circ y$ for all y in L , are also physical, and thus (by the discussion following Theorem 2.8.) certain automorphisms of

\mathcal{L} are to be treated as physical. However, there are automorphisms of \mathcal{L} that are not definable in the appropriate first order language for \mathcal{L} . What of these automorphisms? Are they in any way to be considered as physical entities? In general, answers to questions like these depend on the level of abstraction one allows in the empirical formulation. However, in the present case as well as in others, the following question is a better one to ask: Will it get us into trouble if we assume all automorphisms of \mathcal{L} are physical entities? I believe the answer to this latter question to be definitely 'No': As we will soon see, all automorphisms of \mathcal{L} are meaningful relations of \mathcal{L} , and as such can be added to \mathcal{L} without changing the nature of quantification of \mathcal{L} , and thus their assumption as physical entities cannot change any numerically based physical law. This is not always the case: as we will see, this is not true for automorphisms of qualitative structures that are interval scalable.

Let γ be an endomorphism of \mathcal{Y} . Then γ can be considered as a relation on Y , and as such, γ is automorphism meaningful if and only if for each automorphism α of \mathcal{Y} ,

$$\alpha[\gamma(y)] = \gamma[\alpha(y)]$$

for all y in Y , i.e., if and only if γ commutes with each automorphism of \mathcal{Y} . Similarly, γ is endomorphism meaningful if and only if it commutes with each endomorphism of \mathcal{Y} . Thus it is only in special circumstances such as scalar or monotonic pre-scalar structures that all automorphisms are automorphism meaningful.

Let $S(x, y, u, v)$ hold if and only if x, y, u, v are in Re and $x - y \geq u - v$. Then it is not difficult to show that all automorphisms of $\langle Re, S \rangle$ are of the form $\alpha(z) = rz + s$ for some r in Re^+ and some s in Re , i.e., that $\langle Re, S \rangle$ is interval scalable. Now, the only automorphism of $\langle Re, S \rangle$ that commutes with every other automorphism is the identity, ι , and thus ι is the only automorphism meaningful automorphism of $\langle Re, S \rangle$.

It is easy to show that all relations on Y that are definable in an appropriate first order language for \mathcal{Y} are automorphism meaningful. Thus for the structure $\langle Re, S \rangle$ of the above paragraph, the identity automorphism ι is the only first order definable automorphism.

These considerations are summarized in the following theorem:

THEOREM 3.3. *The following five statements are true:*

1. *Each primitive relation of \mathcal{Y} is endomorphism meaningful.*
2. *Each relation on Y that is first order definable in terms of the primitives of \mathcal{Y} is automorphism meaningful.*
3. *An automorphism of \mathcal{Y} that does not commute with every other automorphism of \mathcal{Y} is not first order definable in terms of the primitives of \mathcal{Y} .*
4. *An endomorphism of \mathcal{Y} is endomorphism meaningful if and only if it commutes with every endomorphism of \mathcal{Y} .*
5. *An automorphism of \mathcal{Y} is automorphism meaningful if and only if it commutes with every automorphism of \mathcal{Y} .*

Although some individual automorphisms of \mathcal{Y} may not be meaningful, the *concept* of automorphism may still be meaningful. This is possible since the concept of automorphism is at a different level of abstraction than individual automorphisms:

Each individual automorphism of \mathcal{Y} is a relation on Y and thus a 'first order concept', while the set of automorphisms of \mathcal{Y} – i.e., the concept of automorphism of \mathcal{Y} – is at a higher and more abstract level and is a 'second order concept'. For the measurement of length, a somewhat analogous situation arises: In the extensive structure for length, $\langle L, \geq, \circ \rangle$, the set of lengths L is a first order concept and is automorphism meaningful, while any particular member of L , say the meter rod in Paris that is used as the basis for the metric measure of length, is at a less abstract level (a 'zeroth order concept') and is not automorphism meaningful.

DEFINITION 3.5. Let \mathcal{S} be a set of relations of Y . Then \mathcal{S} is said to be *automorphism meaningful* if and only if for each relation R on Y and each automorphism α of \mathcal{Y} , R is in \mathcal{S} if and only if $\alpha(R)$ is in \mathcal{S} . \mathcal{S} is said to be *endomorphism meaningful* if and only if for each relation R on Y and each endomorphism γ of \mathcal{Y} , R is in \mathcal{S} if and only if $\gamma(R)$ is in \mathcal{S} . \square

THEOREM 3.4. *The set of automorphisms and endomorphisms of \mathcal{Y} are automorphism meaningful but may not be endomorphism meaningful.*

Proof. It is easy to show that for all automorphisms α and β of \mathcal{Y} and endomorphisms γ of \mathcal{Y} that $\beta(\alpha)$ is an automorphism of \mathcal{Y} and $\beta(\gamma)$ is an endomorphism of \mathcal{Y} , and thus that the sets of automorphisms and endomorphisms of \mathcal{Y} are automorphism meaningful.

If all endomorphisms of \mathcal{Y} are automorphisms of \mathcal{Y} then the sets of automorphisms and endomorphisms of \mathcal{Y} are endomorphism meaningful. However, if γ is an endomorphism of \mathcal{Y} that is not an automorphism, then $\gamma(\iota)$ is not an endomorphism since the domain of $\gamma(\iota)$ is not Y , and thus in this case, sets of automorphisms and endomorphisms of \mathcal{Y} are not endomorphism meaningful. \square

Theorem 3.4. indicates that endomorphism meaningfulness is probably not the correct way of formulating the invariance condition for relations on Y . However, a variety of families of transformations can agree with endomorphism meaningfulness on (the first order) relations on Y but yet disagree on higher order relations. Methods for constructing some of these will now be considered.

CONVENTION. Let γ be a function from a subset of Y into Y , and let S be a n -ary relation on Y for some n . Then, by definition, $\gamma(S)$ is the n -ary relation T on $\gamma(Y)$ such that for each y_1, \dots, y_n in the domain of γ ,

$$S(y_1, \dots, y_n) \text{ iff } T[\gamma(y_1), \dots, \gamma(y_n)].$$

Let β be a function from a subset of Y into Y . Then β can be regarded as binary relation, and as such the $\gamma(\beta)$ is defined by the above. It is easy to show that (i) $\gamma(\beta)$ is a function from a subset of Y into Y , and (ii) an element y of Y is in the domain of $\gamma(\beta)$ if and only if $\beta(y)$ and y are in the domain of γ . It should be noted that (ii) occurs since we consider β as a relation, and that this differs from the usual mathematical use of the notation, $\gamma(\beta)$ which does not require y to be in the domain of γ . Throughout the rest of this section, the notation $\gamma(\beta)$ will be interpreted in the former sense with β considered as a relation.

Throughout the rest of this section, we will also extend the notation for the composition of functions, $*$, as follows: Let f, g be arbitrary functions and let $A = \text{domain } f$ and $B = \text{domain } g$. Then, by definition, $f * g$ is the function h with domain $A \cap g(B)$ such that for each y in $A \cap g(B)$, $h(y) = f[g(y)]$. \square

DEFINITION 3.6. β is said to be *partial endomorphism* of $\mathcal{Y} = \langle Y, R_1, R_2, \dots \rangle$ if and only if β is a function from a subset (possibly empty), of Y into Y such that for each y_1, \dots, y_{n_i} in the domain of β ,

$$R_i(y_1, \dots, y_{n_i}) \text{ iff } R_i[\beta(y_1), \dots, \beta(y_{n_i})]$$

for all i .

A *partial automorphism* of \mathcal{Y} is a partial endomorphism of \mathcal{Y} that is a one-to-one function.

Note that if β is a partial automorphism of \mathcal{Y} , then β^{-1} is a partial automorphism of \mathcal{Y} . Also note by this definition that the *empty partial endomorphism* (the one with empty domain) is a partial automorphism.

β is said to be a *partial identity* of \mathcal{Y} if and only if β is the restriction of the identity automorphism ι of \mathcal{Y} to a subset of Y .

Note that if β is a partial identity of \mathcal{Y} , then in general, $\beta * \beta^{-1} \neq \beta^{-1} * \beta$ since $\beta * \beta^{-1}$ and $\beta^{-1} * \beta$ may have different domains.

Let E be a set of partial endomorphisms of \mathcal{Y} . E is said to be *closed under $*$* if and only if for each α, β in E , $\alpha * \beta$ is in E . E is said to be *closed under E* if and only if for each α, β in E , $\alpha(\beta)$ is in E . Let $E_1 = E$, and for $i \in I^+$, let $E_{i+1} = \{\alpha * \beta \mid \alpha, \beta \in E_i\}$, and let $F = \cup_{i \in I^+} E_i$. Then F is called the *$*$ -closure* of E . Let $E^1 = E$ and for $i \in I^+$, let $E^{i+1} = \{\alpha(\beta) \mid \alpha, \beta \in E_i\}$ and let $H = \cup_{i \in I^+} E^i$. Then H is called the *E -closure* of E .

Let E be a set of partial endomorphisms, S be a n -ary relation on Y , and \mathcal{S} be a set of relations on Y . Then S is said to be *E -meaningful* if and only if for each α in E and each y_1, \dots, y_n the domain of α ,

$$S(y_1, \dots, y_n) \text{ iff } S[\alpha(y_1), \dots, \alpha(y_n)],$$

and \mathcal{S} is said to be *E -meaningful* if and only if for each relation T on Y ,

$$T \in \mathcal{S} \text{ iff } \alpha(T) \in \mathcal{S}.$$

Let E be a set of endomorphisms of \mathcal{Y} . Then the *partial closure* of E is the set E' where

$$E' = \{\alpha \mid \alpha \text{ is a partial endomorphism that is the restriction of } \beta \text{ to } Z \text{ for some } \beta \text{ in } E \text{ and some } Z \subseteq Y\}. \quad \square$$

LEMMA 3.1. *Let E be a nonempty set of endomorphisms of \mathcal{Y} , E' be the closure of E under $*$, S be a n -ary relation on Y , and \mathcal{S} be a nonempty set*

of relations on \mathcal{Y} . Then S is E -meaningful if and only if it is E' -meaningful, and \mathcal{S} is E -meaningful if and only if it is E' -meaningful.

Proof. Since $E' \supseteq E$, E' -meaningfulness implies E -meaningfulness.

Let γ be an element of E' . Then $\gamma = \beta_k * \beta_{k-1} * \dots * \beta_1$ for some β_1, \dots, β_k in E and k in I^+ .

Suppose S is E -meaningful. Then for each y_1, \dots, y_n in the domain of γ ,

$$\begin{aligned} S(y_1, \dots, y_n) & \text{ iff } S[\beta_1(y_1), \dots, \beta_1(y_n)] \\ & \text{ iff } S[\beta_2 * \beta_1(y_1), \dots, \beta_2 * \beta_1(y_n)] \\ & \dots \\ & \text{ iff } S[\beta_k * \dots * \beta_1(y_1), \dots, \beta_k * \dots * \beta_1(y_n)] \\ & \text{ iff } S[\gamma(y_1), \dots, \gamma(y_n)], \end{aligned}$$

and thus S is E' -meaningful.

Suppose \mathcal{S} is E -meaningful. Then for each relation T on Y ,

$$\begin{aligned} T \in \mathcal{S} & \text{ iff } \beta_1(T) \in \mathcal{S} \\ & \text{ iff } \beta_2 * \beta_1(T) \in \mathcal{S} \\ & \dots \\ & \text{ iff } \beta_k * \dots * \beta_1(T) \in \mathcal{S} \\ & \text{ iff } \gamma(T) \in \mathcal{S}, \end{aligned}$$

and thus \mathcal{S} is E' -meaningful. \square

LEMMA 3.2. *Suppose E is a nonempty set of endomorphisms of Y , E' is the partial closure of E , and S is a n -ary relation on Y . Then S is E -meaningful if and only if it is E' -meaningful.*

Proof. Suppose S is E -meaningful. Let γ be an arbitrary element of E' . Let β in E be such that γ is a restriction of β . Then for each y_1, \dots, y_n in the domain of γ ,

$$\begin{aligned} S(y_1, \dots, y_n) & \text{ iff } S[\beta(y_1), \dots, \beta(y_n)] \\ & \text{ iff } S[\gamma(y_1), \dots, \gamma(y_n)], \end{aligned}$$

and thus S is E' -meaningful. Now suppose S is E' -meaningful. Since $E' \supseteq E$, it follows that S is E -meaningful. \square

THEOREM 3.5. *Suppose E is a set of partial endomorphisms of Y that contains the identity automorphism, ι , of \mathcal{Y} . Let F be the E -closure of E . Then E -meaningfulness and F -meaningfulness agree on relations of Y , and F is a F -meaningful set of relations of Y .*

Proof. Let E' be the $*$ closure of E . Then by Lemma 3.1., E -meaningfulness and E' meaningfulness agree on relations of Y . Let E'' be the partial closure of E' . Then it follows from Lemma 3.2. that E and E'' meaningfulness agree on relations of Y . Since for each α, β in E , $\alpha(\beta)$ is a restriction of $\alpha * \beta$, it is easy to show that each member of F is a restriction of a member of E' (since E' is closed under $*$). Thus $E'' \supseteq F \supseteq E$, and from this and the agreement of E and E'' -meaningfulness on relations on Y , it immediately follows that E and F -meaningfulness agree on relations of Y . Now to show that F is a F -meaningful set of relations, we let T be an arbitrary relation on Y . If T is in F , then $\alpha(T)$ is in F for each α in F since F is the E -closure of E . And if $\alpha(T)$ is in F for each α in F , then in particular, $\iota(T) = T$ is in F . \square

Applying Theorem 3.5. to the set of endomorphisms of \mathcal{Y} , we see that this set can be extended to a reasonable set of F of partial endomorphisms that is meaningful with respect to itself and agrees with endomorphism meaningfulness on relations of Y . This seems to suggest to me that in this case F -meaningfulness is probably a superior measurement-theoretic concept to endomorphism meaningfulness, especially if one is considering meaningfulness of higher order concepts.

There are other ways of extending sets of partial endomorphisms without altering the meaningfulness of relations on Y . The following lemma gives an important one:

LEMMA 3.3. *Suppose B is a nonempty set of partial automorphisms of \mathcal{Y} and S is a n -ary relation on Y and \mathcal{S} is a nonempty set of relations on \mathcal{Y} . Let $C = B \cup \{\alpha^{-1} \mid \alpha \in B\}$. Then S is B -meaningful if and only if it is C -meaningful, and \mathcal{S} is B -meaningful if and only if it is C meaningful.*

Proof. Suppose S is C -meaningful. Since $C \supseteq B$, it immediately follows that S is B -meaningful. Now suppose S is B -meaningful. Let α be an arbitrary element of C . We need only show that for all y_1, \dots, y_n in the domain of α ,

$$S(y_1, \dots, y_n) \quad \text{iff} \quad S[\alpha(y_1), \dots, \alpha(y_n)].$$

If $\alpha \in B$, this immediately follows from B -meaningfulness. If $\alpha \notin B$, then α^{-1} is in B and by B -meaningfulness, for each x_1, \dots, x_n in the domain of α^{-1} ,

$$(3.4) \quad S(x_1, \dots, x_n) \text{ iff } S[\alpha^{-1}(x_1), \dots, \alpha^{-1}(x_n)],$$

and letting $y_1 = \alpha^{-1}(x_1), \dots, y_n = \alpha^{-1}(x_n)$, we see that Equation (3.4) becomes

$$(3.5) \quad S[\alpha(y_1), \dots, \alpha(y_n)] \text{ iff } S(y_1, \dots, y_n),$$

and Equation (3.5) holds for each z_i that is $\alpha^{-1}(u_i)$ for some u_i in the domain of α^{-1} . Thus we have shown S to be C -meaningful.

If \mathcal{S} is C -meaningful, it immediately follows it is B -meaningful. Suppose \mathcal{S} is B -meaningful. Then for all relations T on X and all α in B ,

$$\begin{aligned} T \in \mathcal{S} &\text{ iff } \alpha * \alpha^{-1}(T) \in \mathcal{S} \\ &\text{ iff } \alpha^{-1}(T) \in \mathcal{S}. \quad \square \end{aligned}$$

DEFINITION 3.7. $\langle B, * \rangle$ is said to be a *pseudo-group of partial automorphisms of \mathcal{Y}* if and only if B is a nonempty set of partial automorphisms of \mathcal{Y} such that for each α, β in B , $\alpha * \beta$ and α^{-1} are in B . If $\langle B, * \rangle$ is a pseudo-group of automorphisms of \mathcal{Y} , $\langle B, * \rangle$ is said to be *meaningful* if and only if ι is in B and if α, β are in B , then $\alpha(\beta)$ is in B .

Let C be a nonempty set of partial automorphisms. $\langle B, * \rangle$ is said to be the *pseudo-group generated by C* if and only if (i) $\langle B, * \rangle$ is a pseudo-group and $B \supseteq C$, and (ii) for each pseudo-group $\langle D, * \rangle$, if $D \supseteq C$, then $D \supseteq B$. The analogous definition for *meaningful pseudo-group generated by C* is given in the obvious way. \square

Let C be a nonempty set of partial automorphisms of \mathcal{Y} . Then it easily follows that the pseudo-group generated by C and the meaningful pseudo-group generated by C exist.

LEMMA 3.4. *Let B be a nonempty set of partial automorphisms of \mathcal{Y} , $\langle B', * \rangle$ the pseudo-group generated by B , S a n -ary relation on Y , and \mathcal{S} a nonempty set of relations on Y . Suppose B is closed under $*$. Then S is B -meaningful if and only if it is B' -meaningful, and \mathcal{S} is B -meaningful if and only if it is B' meaningful.*

Proof. Let $B_1 = B$ and $C_1 = \{\alpha^{-1} \mid \alpha \in B_1\}$. Suppose i is an element of I^+ and B_i and C_i have been defined. Let B_{i+1} be the $*$ -closure of $B_i \cup C_i$ and $C_{i+1} = \{\alpha^{-1} \mid \alpha \in B_{i+1}\}$. Then it easily follows from Lemmas 3.3. and 3.1. that S and \mathcal{S} are B_{i+1} -meaningful if and only if they are B_i -meaningful. Thus for each i in I^+ , S and \mathcal{S} are B_i -meaningful if and only if they are B -meaningful. From this and $B' = \cup_{i \in I^+} B_i$, it then follows that S and \mathcal{S} are B -meaningful if and only if they are B' -meaningful. \square

THEOREM 3.6. *Suppose B is a nonempty set of partial automorphisms of \mathcal{Y} , S is an arbitrary n -ary relation on Y , and \mathcal{S} an arbitrary nonempty set of relations on Y . Let $\langle B', * \rangle$ be the pseudogroup generated by B and $\langle B'', * \rangle$ the meaningful pseudo-group generated by B . Then (i) B and B' -meaningfulness coincide for S and \mathcal{S} , and (ii) B and B'' -meaningfulness coincide for S , and if B contains an automorphism of \mathcal{Y} , then B'' is B'' -meaningful.*

Proof. (i) follows from Lemma 3.4. and (ii) by an argument similar to the proof of Theorem 3.5. \square

The following example illustrates the usefulness of the above concepts.

EXAMPLE 3.3. Let $\mathcal{R}_1 = \langle Re^+, \geq, + \rangle$. Then it can be shown that multiplications by positive reals are the automorphisms of \mathcal{R}_1 , and that all endomorphisms of \mathcal{R}_1 are automorphisms.

Let $\mathcal{R}_2 = \langle (0, 1), \geq, + \rangle$, where '(0, 1)' denotes the open interval of the reals with end points 0 and 1. Then it can be shown that the identity is the only automorphism of \mathcal{R}_2 , and that multiplications by positive reals ≤ 1 are the endomorphisms of \mathcal{R}_2 . Let E be the set of endomorphisms of \mathcal{R}_2 . Then each element of E is a partial automorphism of \mathcal{R}_2 . Let E' be the pseudo group generated by E . Then it can be shown that

$$E' = \{\alpha \mid \text{there exists } r \text{ in } (0, 1) \text{ such that } (0, r) = \text{domain } \alpha \\ \text{and for all } x \text{ in } (0, r), \alpha(x) = sx \text{ for some } s \text{ in } Re^+ \text{ such} \\ \text{that } sr < 1.\}$$

Furthermore, it can also be shown that E' is also the meaningful pseudo group generated by E .

Let $\mathcal{R}_3 = \langle (2, 10), \geq, + \rangle$. Then it can be shown that the identity is the only endomorphism (and therefore the only automorphism) of \mathcal{R}_3 . For each r in $(1/5, 5)$, let α_r be the following function:

$$\text{domain } \alpha = \begin{cases} (2/r, 10) & \text{if } r \text{ is in } (1/5, 1] \\ (2, 10/r) & \text{if } r \text{ is in } [1, 5) \end{cases}$$

and $\alpha_r(x) = rx$ for each x in the domain of α_r . Let $B = \{\alpha_r \mid r \text{ is in } (1/5, 5)\}$. Then for each α in B , α^{-1} is in B , and in fact, $\alpha^{-1} = \alpha_{1/r}$ where r is such that $\alpha = \alpha_r$. However, B is not a pseudo group since it is not closed under $*$. (This can be seen by example: $\alpha_{1/4} * \alpha_3$ has domain $(8/3, 10/3)$ and is therefore not in B .) It also can be shown that the pseudo group generated by B coincides with the meaningful pseudo group generated by B . \square

The concept of meaningfulness arose in measurement theory through concerns about the justifications of certain statistical procedures; later (Luce, 1978) it was modified to more clearly explicate certain procedures in dimensional analysis. But the basic idea that invariants under natural sets of transformations have a special status goes back much further to a very famous and influential paper by Felix Klein, "Vergleichende Betrachtungen über neuere geometrische Forschungen", published at Erlangen in 1872. Klein proposed that geometrical properties were exactly those that were left invariant by the automorphisms of the geometrical space. Subsequently, Klein and his followers were able to describe all the known geometries in terms of this automorphism invariance concept. This method of dealing with geometry became known as the Erlanger Program. And for a time when new geometries, such as the space-time geometry inherent in Einstein's theory of special relativity, were introduced into mathematics, the Erlanger Program was extended to encompass these as well. However, the Erlanger Program was not able to encompass the space-time geometry inherent in Einstein's general theory of relativity since space-time in this geometry has the identity as its only automorphism. Some attempts were made by followers of Klein to save the Erlanger Program by considering other invariance concepts (which by my reading of the literature appears to be invariance under something like a pseudogroup of partial automorphisms, but I have been unable to find an absolutely clear statement of this), but these attempts have had very little influence, perhaps in part because of a shift in emphasis in mathematics to a more algebraic approach to geometry.

In this section, a number of concepts of qualitative meaningfulness have been presented, and the problem remains of deciding which, if any, is the 'correct' concept. It is my view that there is no single correct concept of

meaningfulness. I believe that in the final analysis the choice of the ‘correct’ invariance concept for a structure will not be determined solely by the structure, but in general will depend upon features of the intended measurement application. What we have today are a handful of successful applications of the various meaningfulness concepts; what is still lacking is a general theory of invariance and inference based upon invariance. The meaningfulness concepts presented above are attempts to abstract the common core of this handful of successful invariance applications, and are not based upon any detailed philosophical analysis, and thus their usefulness and generality are somewhat in doubt. Hopefully in the future someone will find a more direct and comprehensive approach to this important problem.

Of the invariance concepts considered in this section, automorphism meaningfulness has the greatest applicability, mainly because the most important structures that appear in measurement have an abundance of automorphisms. Endomorphism and qualitative \mathcal{N} -meaningfulness, when they do not coincide with automorphism meaningfulness, thus far have had far fewer applications. I also believe that these latter two concepts have inherent difficulties, which arise from the fact that representations of the qualitative structure are only required to be *into* (rather than *onto*) the representing structure. Interesting enough, it is this ‘into’ property of representations that make endomorphism and qualitative \mathcal{N} -meaningfulness natural concepts for measurement. (This can be seen by considering the case of qualitative structure \mathcal{Y} that has a one-to-one endomorphism that is not an automorphism. Then if for some ϕ , \mathcal{N} , and α , if ϕ is a \mathcal{N} -representation for \mathcal{Y} and α is a one-to-one endomorphism of \mathcal{Y} that is not an automorphism, then $\phi\alpha$ is a \mathcal{N} -representation of \mathcal{Y} that is into but not onto \mathcal{N} .) However, to my knowledge, the practice of using ‘into’ representations for the general measurement case has never been philosophically justified. The situations where ‘into’ representations have been useful are rather special and are characterized by conditions similar to compatibility (Definition 2.3.), or as I prefer to see it, characterized by the representing structure being isomorphic to an extension of the qualitative structure where the automorphisms (or in some situations certain key partial automorphisms) of the qualitative structure extend to automorphisms of the extension. It is my belief that measurement of general structures should be based either upon representations that are isomorphisms onto the representing structure, or, if the situation

demands it, upon an appropriate set partial isomorphisms. (A *partial isomorphism* is an isomorphism of a restriction of the qualitative structure into the representing structure.) This should be done in a way so that the corresponding qualitative concept is either automorphism meaningfulness or H -meaningfulness for some meaningful subgroup of partial automorphisms. The consequences of these suggestions for current measurement structures are minimal; however their effect upon bringing new types of structures and techniques into the measurement arena could be significant, particularly for structures that resemble those in general relativity and differential geometry.

4. SCALAR PRODUCT STRUCTURES

Physical structures are the best known and understood empirical structures which are decomposable into well-defined empirical substructures. In this case, they are decomposable into their various physical dimensions, e.g., length, mass, time, which qualitatively can be assumed to be extensive structures. In this section, generalizations of this physical case will be considered. To simplify notation and exposition, only the case of structures decomposable into two components will be considered, but the concepts and results of this section are easily extendable to the general multicomponent case.

CONVENTION. Throughout the rest of this section, unless explicitly stated otherwise, let \succsim be a weak ordering on the nonempty set $X \times P$. It will be assumed throughout this section, unless explicitly stated otherwise, that $\langle X \times P, \succsim \rangle$ satisfies the following three conditions:

- (i) *Unrestricted solvability*: for each xp, yq in $X \times P$, there exist z, r such that $xp \sim yr$ and $xp \sim zq$;
- (ii) *Independence*: for each x, y in X , if for some b in P , $xb \succsim yb$, then for all r in P , $xr \succsim yr$; and for each p, q in P , if for some a in X , $ap \succsim aq$, then for all z in X , $zp \succsim zq$.
- (iii) *Density*: for each x, y in X , if for some b in P , $xb \succ yb$, then for some z in X , $xb \succ zb \succ yb$.

It easily follows from unrestricted solvability and independence that the binary relations \succeq_X and \succeq_P defined on X and P respectively as follows are weak orderings: for each x, y in X ,

$$x \succeq_X y \quad \text{iff} \quad \text{for some } b \text{ in } P, xb \succsim yb,$$

and for each p, q in P ,

$$p \succeq_P q \quad \text{iff} \quad \text{for some } a \text{ in } X, ap \succsim aq.$$

To simplify notation and exposition, it will be assumed throughout this section, unless explicitly stated otherwise, that the following condition holds:

- (iv) \succeq_X and \succeq_P are total orderings on X and P respectively. \square

Independence and unrestricted solvability give $\langle X \times P, \succsim \rangle$ enough structure to allow monotonic increasing operations to be defined on $\langle X, \succeq_X \rangle$ and $\langle P, \succeq_P \rangle$, and thus give it the potential to be the basis of a ratio scale. This will be shown in detail in Definitions 5.1., 5.2., and 5.3. and Theorem 5.6.

CONVENTION. Throughout this section, unless explicitly stated otherwise, let

$$\mathcal{E} = \langle X \times P, \succsim, R_1, R_2, \dots, S_1, S_2, \dots, T_1, T_2, \dots \rangle,$$

where R_1, R_2, \dots are relations on X , S_1, S_2, \dots are relations on P , and T_1, T_2, \dots are relations on $X \times P$. Let $\mathcal{L} = \langle X, \succeq_X, R_1, R_2, \dots \rangle$ and $\mathcal{P} = \langle P, \succeq_P, S_1, S_2, \dots \rangle$. \square

DEFINITION 4.1. Recall that in Section 2, the results about scalar structures do not depend in an essential way upon the particular functions and relations in the qualitative structure. Thus for the current setup if we assume that \mathcal{L} and \mathcal{P} satisfy density, homogeneity, and automorphism commutivity of Definition 2.5. we may call them ‘scalar structures’. Furthermore, by the remarks following Theorem 2.7., if \mathcal{L} is an Archimedean scalar structure then there exists a function ϕ from X into Re^+ such that for some numerical structure \mathcal{N} , ϕ is an isomorphism onto \mathcal{N} and the set of automorphisms of \mathcal{N} is a subset of all multiplications by positive reals. Such a function ϕ will be called a *scalar representation* for \mathcal{L} . An analogous definition for a *scalar representation* for \mathcal{P} holds. \square

The following two definitions give important ways in which the components of \mathcal{E} may be related. The first is a quantitative way and the second is a qualitative way. Later theorems will show that in many important cases, these ways coincide.

DEFINITION 4.2. $\langle \phi, \psi \rangle$ is said to be a *product representation* for \mathcal{E} if and only if $\phi: X \rightarrow Re^+$, $\psi: P \rightarrow Re^+$, and for each xp, yq in $X \times P$,

$$xp \succsim yq \quad \text{iff} \quad \phi(x)\psi(p) \geq \phi(y)\psi(q). \quad \square$$

DEFINITION 4.3. \circ is said to be an *X-distributive operation* of \mathcal{E} if and only if \circ is a binary operation on X and for each x, y, u, v in X and each p, q in P , if $xp \sim uq$ and $yp \sim vq$, then $(x \circ y)p \sim (u \circ v)q$. \square

DEFINITION 4.4. For the rest of this section, let x_0p_0 be a fixed element of $X \times P$ and let ξ and τ be functions defined as follows: $\xi: X \times P \rightarrow X$, $\tau: X \rightarrow P$, and for each xp in $X \times P$,

$$\xi(xp)p_0 = xp$$

and

$$xp_0 = x_0\tau(x).$$

(Unrestricted solvability guarantees that ξ and τ are well defined by this procedure.) \square

DEFINITION 4.5. For each x in X define the function β_x on X as follows: for each y in X ,

$$\beta_x(y) = \xi[y\tau(x)]. \quad \square$$

LEMMA 4.1. *Suppose \circ is a X-distributive operation of \mathcal{E} . Then for each x in X , β_x is an automorphism of the structure $\langle X, \succeq_X, \circ \rangle$.*

Proof. Let x, y , and z be arbitrary elements of X .

1. Suppose $y \succeq_X z$. Then by independence

$$y\tau(x) \succsim z\tau(x),$$

and thus

$$\xi[y\tau(x)]p_0 \succsim \xi[z\tau(x)]p_0,$$

which by independence yields

and thus $\xi[y\tau(x)] \succeq_X \xi[z\tau(x)],$
 $\beta_x(y) \succeq_X \beta_x(z).$

2. Since

and $\beta_x(y)p_0 = \xi[y\tau(x)]p_0 \sim y\tau(x)$
 $\beta_x(z)p_0 = \xi[z\tau(x)]p_0 \sim z\tau(x),$

it follows from X -distributivity that

$$[\beta_x(y) \circ \beta_x(z)]p_0 \sim (y \circ z)\tau(x),$$

and thus that

$$\beta_x(y) \circ \beta_x(z) = \xi[(y \circ z)\tau(x)] = \beta_x(y \circ z).$$

3. To show β_x is onto X , we need only show for some u , $\beta_x(u) = y$. By unrestricted solvability, let u be such that $u\tau(x) \sim yp_0$. Then $\beta_x(u) = \xi[u\tau(x)] = y$. \square

LEMMA 4.2. *Suppose \mathcal{L} is homogeneous, \circ is a X -distributive operation of \mathcal{C} , $\langle X, \succeq_X, \circ \rangle$ is a positive concatenation structure, and \circ and \succeq_X are automorphism meaningful relations for the structure \mathcal{L} . Then the automorphisms of \mathcal{L} and $\langle X, \succeq_X, \circ \rangle$ coincide.*

Proof. Let A be the set of automorphisms of \mathcal{L} and B the set of automorphisms of $\langle X, \succeq_X, \circ \rangle$. Since \circ and \succeq_X are automorphism meaningful relations of \mathcal{L} , each element of A is an automorphism of $\langle X, \succeq_X, \circ \rangle$, i.e., $A \subseteq B$. Let β be an arbitrary element of B . Let x be an arbitrary element of X . Since \mathcal{L} is homogeneous, let α in A be such that $\alpha(x) = \beta(x)$. Then $\alpha^{-1} * \beta$ is an element of B and $(\alpha^{-1} * \beta)(x) = x$. Theorem 2.1. of Cohen and Narens (1979) shows that in this case $\alpha^{-1} * \beta$ must be the identity, ι . Thus $\alpha = \beta$. Since β is an arbitrary element of B , it follows that $B \subseteq A$. Thus $B = A$. \square

THEOREM 4.1. *Suppose \mathcal{L} is a scalar structure, \circ is a X -distributive operation of \mathcal{C} , $\langle X, \succeq_X, \circ \rangle$ is a positive concatenation structure and \succeq_X and \circ are automorphism meaningful relations for the structure \mathcal{L} . Then for each scalar representation ϕ of \mathcal{L} there exists ψ such that $\langle \phi, \psi \rangle$ is a product representation for \mathcal{C} .*

Proof. Let $\mathcal{L}' = \langle X, \succeq_X, R_1, R_2, \dots \rangle$ where $\mathcal{L} = \langle X, R_1, R_2, \dots \rangle$. Since \succeq_X is an automorphism meaningful relation for \mathcal{L} , the automorphisms of \mathcal{L} and \mathcal{L}' agree. Let A be the set of automorphisms of \mathcal{L}' . Then by Lemma 4.2., A is the set of automorphisms of $\langle X, \succeq_X, \circ \rangle$. Let \succeq_X be the ordering defined on A by: for each α, β in A , $\alpha \succeq_X \beta$ if and only if for some x in X , $\alpha(x) \succeq_X \beta(x)$. Then by Theorem 2.4. of Cohen and Narens (1979), $\langle A, \succeq_X, * \rangle$ is an Archimedean totally ordered group of automorphisms of $\langle X, \succeq_X, \circ \rangle$. Therefore, \mathcal{L}' is an Archimedean scalar structure (where the 'Archimedeaness' of A is defined in terms of \succeq_X). Let ϕ be an arbitrary scalar representation for \mathcal{L}' . Since \mathcal{L}' and $\phi(\mathcal{L}')$ are isomorphic, we may identify them, and under this identification ϕ may be considered as the identity ι , and thus the theorem follows by showing there exists ψ such that $\langle \iota, \psi \rangle$ is a product representation for \mathcal{L} .

By unrestricted solvability, $\tau^{-1}(p)$ exists for each p in P . Let $\psi = \tau^{-1}$. Then ψ is a function from P onto X , and since ι is a scalar representation \mathcal{L}' , ψ is into Re^+ .

Note that for each x in X ,

$$(4.1) \quad \beta_x(x_0) = \xi[x_0\tau(x)] = x.$$

Also note that since ι is a scalar representation for \mathcal{L}' , all automorphisms of \mathcal{L}' are multiplications by positive reals, and thus by Lemmas 4.1., 4.2., and Equation (4.1), β_x is multiplication by x/x_0 .

Let xp, yq be arbitrary elements of $X \times P$. Then

$$\begin{aligned} xp \succsim yq & \text{ iff } \xi(xp) \succeq_X \xi(yq) \\ & \text{ iff } \xi(x\tau[\tau^{-1}(p)]) \succeq_X \xi(y\tau[\tau^{-1}(q)]) \\ & \text{ iff } \xi(x\tau[\psi(p)]) \succeq_X \xi(y\tau[\psi(q)]) \\ & \text{ iff } \beta_{\psi(p)}(x) \succeq_X \beta_{\psi(q)}(y) \\ & \text{ iff } \frac{\psi(p)}{x_0} \cdot x \geq \frac{\psi(q)}{x_0} \cdot y \\ & \text{ iff } \iota(x)\psi(p) \geq \iota(y)\psi(q). \quad \square \end{aligned}$$

An important part of the proof of Theorem 4.1. is that β_x is an automorphism of \mathcal{L} for each x in X . This condition is derived from Lemmas 4.1. and 4.2. Lemma 4.2. assumes that $\langle X, \succeq_X, \circ \rangle$ is a positive concatenation

structure. This assumption is made so that $\langle X, \succeq_X, \circ \rangle$ will not have too many automorphisms, and any other set of assumptions that require it to satisfy one point uniqueness for automorphisms (if $\alpha(x) = \beta(x)$ for some automorphisms α, β of $\langle X, \succeq_X, \circ \rangle$ and some x in X , then $\alpha = \beta$) will do in place of assuming $\langle X, \succeq_X, \circ \rangle$ to be a positive concatenation structure.

The assumption in Theorem 4.1. that \succeq_X is an automorphism meaningful relation of \mathcal{L} is very natural, as the following theorem shows:

THEOREM 4.2. *Suppose \mathcal{L} is a scalar structure, ϕ is a scalar representation for \mathcal{L} , and $\langle \phi, \psi \rangle$ is a product representation for \mathcal{C} . Then \succeq_X is automorphism meaningful for the structure \mathcal{L} .*

Proof. Let x, y be arbitrary elements of X and α be an arbitrary automorphism of \mathcal{L} . Then, since ϕ is a scalar representation of \mathcal{L} , $\phi(\alpha)$ is an automorphism of $\phi(\mathcal{L})$ and thus is multiplication by a positive real r . Thus

$$\begin{aligned} x \succeq_X y & \text{ iff } xp_0 \succeq yp_0 \\ & \text{ iff } \phi(x)\psi(p_0) \geq \phi(y)\psi(p_0) \\ & \text{ iff } \phi(x) \geq \phi(y) \\ & \text{ iff } r\phi(x) \geq r\phi(y) \\ & \text{ iff } \alpha(x) \succeq_X \alpha(y). \end{aligned}$$

Since α, x , and y are arbitrary, it follows that \succeq_X is an automorphism meaningful relation of \mathcal{L} . \square

There often exists a strong relationship between X -distributivity and automorphism meaningfulness, and this relationship is investigated in the next lemma and theorem:

LEMMA 4.3. *Assume the hypotheses of Lemma 4.2. Suppose \oplus is a X -distributive operation of \mathcal{C} . Then \oplus is an automorphism meaningful relation of \mathcal{L} .*

Proof. By Lemmas 4.1. and 4.2., β_x is an automorphism of \mathcal{L} for each x in X . Then for each automorphism α of \mathcal{L} ,

$$\beta_{\alpha(x_0)}(x_0) = \xi(x_0\tau[\alpha(x_0)]) = \alpha(x_0),$$

and thus

$$\alpha^{-1}\beta_{\alpha(x_0)}(x_0) = x_0,$$

which by Theorem 2.1. of Cohen and Narens (1979) shows that $\alpha^{-1}\beta_{\alpha(x_0)} = \iota$ and thus that $\alpha = \beta_{\alpha(x_0)}$. Therefore $\{\beta_x \mid x \in X\}$ is the set of automorphisms of \mathcal{L} , and thus by Lemma 4.1., \oplus is an automorphism meaningful relation of \mathcal{L} . \square

THEOREM 4.3. *Suppose \mathcal{L} is a scalar structure, \mathcal{C} has a product representation $\langle \phi, \psi \rangle$ where ϕ is a scalar representation for \mathcal{L} , $\langle X \times P, \succsim \rangle$ is Dedekind complete, and \oplus is a binary operation on X . Then the following two statements are equivalent:*

- (i) \oplus is a X -distributive operation of \mathcal{C} ;
- (ii) \oplus is an automorphism meaningful relation of \mathcal{L} .

Proof. Since $\langle X \times P, \succsim \rangle$ is Dedekind complete, it easily follows that $\langle X, \succeq_X \rangle$ is Dedekind complete. Let $\langle \phi, \psi \rangle$ be a product representation for \mathcal{C} where ϕ is a scalar representation for \mathcal{L} . Since \mathcal{L} and $\phi(\mathcal{L})$ are isomorphic, we may identify them without losing generality. In this case ϕ is the identity, ι , and each automorphism of \mathcal{L} is multiplication by a positive real. Since multiplications by positive reals are automorphisms of \mathcal{L} , X has arbitrarily large and small elements, which by the Dedekind completeness of $\langle X, \succeq_X \rangle$ implies $X = Re^+$, which in turn by the homogeneity of \mathcal{L} implies that all multiplications by positive reals are automorphisms of \mathcal{L} . It is also immediate that \succeq_X is \geq .

(i) Suppose \oplus is a X -distributive operation of \mathcal{C} . We will first show that $+$ is a X -distributive operation of \mathcal{C} . Suppose x, y, u, v are arbitrary elements of X and p, q are arbitrary elements of P such that $xp \sim uq$ and $yp \sim vq$. Then, since $\langle \iota, \psi \rangle$ is a product representation for \mathcal{C} ,

$$x \cdot \psi(p) = u \cdot \psi(q) \quad \text{and} \quad y \cdot \psi(p) = v \cdot \psi(q).$$

Thus

$$(x + y) \cdot \psi(p) = (u + v) \cdot \psi(q),$$

and therefore $(x + y)p \sim (u + v)q$, i.e., $+$ is X -distributive. Furthermore $\langle X, \succeq_X, + \rangle = \langle Re^+, \geq, + \rangle$ is a positive concatenation structure, and since automorphisms of \mathcal{L} are multiplications by positive reals, \succeq_X and $+$ are automorphism meaningful relations of \mathcal{L} . Thus the hypotheses of

Lemma 4.3. are satisfied, and thus by Lemma 4.3., \oplus is an automorphism meaningful relation of \mathcal{L} .

(ii) Suppose \oplus is an automorphism meaningful relation of \mathcal{L} . Let x, y, u, v be arbitrary elements of X and p, q be arbitrary elements of P such that $xp \sim uq$ and $yp \sim vq$. Then since $\langle \iota, \psi \rangle$ is a product representation for \mathcal{C} ,

$$(4.2) \quad x \cdot \psi(p) = u \cdot \psi(q) \quad \text{and} \quad y \cdot \psi(p) = v \cdot \psi(q).$$

Since each positive real is an automorphism of \mathcal{L} , by the automorphism meaningfulness of \oplus ,

$$(4.3) \quad (x \oplus y) \cdot \psi(p) = [x \cdot \psi(p)] \oplus [y \cdot \psi(p)],$$

and

$$(4.4) \quad (u \oplus v) \cdot \psi(q) = [u \cdot \psi(q)] \oplus [v \cdot \psi(q)].$$

Thus by Equations (4.2), (4.3), and (4.4),

$$(x \oplus y) \cdot \psi(p) = (u \oplus v) \cdot \psi(q),$$

which, since $\langle \iota, \psi \rangle$ is a product representation for \mathcal{C} , yields

$$(x \oplus y)p \sim (u \oplus v)q,$$

and thus that \oplus is a X -distributive operation of \mathcal{E} . \square

The next theorem describes the relationship between product representations of \mathcal{C} .

THEOREM 4.4. *Suppose $\langle \phi, \psi \rangle$ and $\langle \phi_1, \psi_1 \rangle$ are product representations for \mathcal{C} . Then for some r, s, t in Re^+ , $\phi_1 = s\phi^r$ and $\psi_1 = t\psi^r$.*

Proof. Define \circ on $X \times P$ as follows: for each xp, yq in $X \times P$ (by unrestricted solvability) let u in X and v in P be such that $xp \sim up_0$ and $yq \sim x_0v$, and let $xp \circ yq = uv$. Let $\phi' = (1/\phi(x_0))\phi$, $\psi' = (1/\psi(p_0))\psi$, $\phi'_1 = (1/\phi_1(x_0))\phi_1$, and $\psi'_1 = 1/(\psi_1(p_0))\psi$. Then $\langle \phi', \psi' \rangle$ and $\langle \phi'_1, \psi'_1 \rangle$ are product representations for \mathcal{C} , and $\phi'(x_0) = \psi'(p_0) = \phi'_1(x_0) = \psi'_1(p_0) = 1$. Define F and G on $X \times P$ as follows: for each xp in $X \times P$, $F(xp) = \phi'(x)\psi'(p)$ and $G(xp) = \phi'_1(x)\psi'_1(p)$. Then for each xp, yq in $X \times P$,

$$(i) \quad xp \succsim yq \quad \text{iff} \quad F(xp) \geq F(yq) \\ \text{iff} \quad G(xp) \geq G(yq),$$

and letting u, v be such that $xp \sim up_0$ and $yq \sim x_0v$, we get

$$\begin{aligned}
 \text{(ii)} \quad F(xp \circ yq) &= F(uv) \\
 &= \phi'(u)\psi'(v) \\
 &= [\phi'(u)\psi'(p_0)] \cdot [\phi'(x_0)\psi'(v)] \\
 &= [\phi'(x)\psi'(p)] \cdot [\phi'(y)\psi'(q)] \\
 &= F(xp) \cdot F(yq),
 \end{aligned}$$

and similarly,

$$\text{(iii)} \quad G(xp \circ yq) = G(xp) \cdot G(yq).$$

Using (i), (ii), and (iii) it easily follows that $\mathcal{E} = \langle X \times P, \succsim, \circ \rangle$ is a closed extensive structure as defined in Definition 3.1. of Krantz *et al.* (1971) and that F and G are multiplicative representations for \mathcal{E} , which by the ratio scalability of \mathcal{E} (Theorem 3.1. of Krantz *et al.*, 1971) means that we can let r in Re^+ be such that $G = F^r$. Then for each x in X ,

$$\phi'_1(x) = \phi'_1(x)\psi'_1(p_0) = [\phi'(x)]^r \cdot [\psi'(p_0)]^r = [\phi'(x)]^r,$$

and similarly, for each p in P , $\psi'_1(p) = [\psi'(p)]^r$. Thus letting $s = \phi(x_0)^r$ and $t = \psi(p_0)^r$, we get $\phi_1 = s\phi^r$ and $\psi_1 = t\psi^r$. \square

DEFINITION 4.6. \mathcal{E} is said to satisfy *component \mathcal{L} -invariance* if and only if for each automorphism α of \mathcal{L} and each xp, yq in $X \times P$, $xp \sim yq$ iff $\alpha(x)p \sim \alpha(y)q$. An analogous definition holds for *component \mathcal{P} -invariance*. \square

In the physical case, the component invariances of Definition 4.6. is often called dimensional invariance and is an important property of physical structures. The key idea behind dimensional analysis in physics is that complex physical quantities can be factored into basic dimensions in a way that changes in scale of the measurements of the basic dimensions produce changes in scale of the measurement of complex quantities. The following theorem shows that component invariance is closely related to the previous development.

THEOREM 4.5. *Suppose \mathcal{L} is a scalar structure and \oplus is a binary operation on X that is an automorphism meaningful relation of \mathcal{L} . Also suppose that \mathcal{E} satisfies component \mathcal{L} -invariance and \oplus satisfies the following two conditions:*

- (i) *Solvability: for each x, y in X , if $x \succ_X y$, then there are exactly one u and one v such that $x = y \oplus u = v \oplus y$.*
- (ii) *Positivity: for each x, y in X , $x \oplus y \succ_X x$ and $x \oplus y \succ_X y$.*

Then \oplus is a \mathcal{L} -distributive operation of \mathcal{E} .

Proof. Suppose $xp \sim uq$ and $yp \sim vq$. To show the theorem, we need only show that $(x \oplus y)p \sim (u \oplus v)q$. By positivity, solvability, and unrestricted solvability, let u', v' in X be such that

$$(x \oplus y)p \sim (u' \oplus v)q \sim (u \oplus v')q.$$

Thus we need only to show that $v = v'$. By homogeneity in \mathcal{L} , let α be an automorphism of \mathcal{L} such that $\alpha(x) = u$. We will show that $\alpha(y) = v$. Again by homogeneity in \mathcal{L} , let β be an automorphism of \mathcal{L} such that $\beta(x \oplus y) = x$. Then by component \mathcal{L} -invariance,

$$\beta(x \oplus y)p \sim \beta(u \oplus v')q.$$

Thus

$$uq \sim xp \sim \beta(x \oplus y)p \sim \beta(u \oplus v')q,$$

and therefore,

$$u = \beta(u \oplus v').$$

Since \mathcal{L} is a scalar structure, the automorphisms of \mathcal{L} commute, and thus

$$\beta\alpha(x \oplus y) = \alpha\beta(x \oplus y) = \alpha(x) = u = \beta(u \oplus v'),$$

which by the automorphism meaningfulness of \oplus yields

$$\alpha(x) \oplus \alpha(y) = u \oplus \alpha(y) = u \oplus v',$$

and from this and solvability it follows that

$$\alpha(y) = v'.$$

By a similar argument, we can choose an automorphism γ of \mathcal{L} such that $\gamma(y) = v$ and show that $\gamma(x) = u'$. Now by the automorphism meaningfulness of \oplus ,

$$\begin{aligned} \gamma(x \oplus y) &= \gamma(x) \oplus \gamma(y) = u' \oplus v = u \oplus v' \\ &= \alpha(x) \oplus \alpha(y) = \alpha(x \oplus y), \end{aligned}$$

and thus by Lemma 2.1., $\gamma = \alpha$. Therefore

$$v = \gamma(y) = \alpha(y) = v'. \quad \square$$

5. ADDITIONAL THEOREMS AND PROOFS

In this section some new theorems will be shown and the proofs for some theorems of the previous sections will be given. The proofs of Theorems 5.1. and 5.2. assume some familiarity with the basic concepts of nonstandard analysis (e.g., in Robinson, 1966), and Theorems 5.3. and 5.4. rely heavily on results of Cohen and Narens (1979). Theorems 5.5. and 5.6. use concepts developed in Narens and Luce (1976) and Cohen and Narens (1979).

THEOREM 5.1. (*Theorem 2.2.*) *Suppose $\mathcal{Y} = \langle Y, T_j \rangle_{j \in J}$ is a relational structure and S is a set of one-to-one endomorphisms of \mathcal{Y} that commute with one another. Then there exists an extension \mathcal{Y}' of \mathcal{Y} such that each endomorphism in S extends to an automorphism of \mathcal{Y}' .*

Proof by nonstandard analysis. Let \mathcal{E} be an enlargement of \mathcal{Y} . Then ${}^*\mathcal{Y} = \langle {}^*Y, {}^*T_j \rangle_{j \in {}^*J}$. Let W be an internal * finite subset of *S such that $S \subseteq W \subseteq {}^*S$. Let ω be an element of ${}^*I^+$ and $\alpha_1, \dots, \alpha_\omega$ be such that $W = \{\alpha_1, \dots, \alpha_\omega\}$ is an internal indexing of W . Let ν be an infinite positive integer and $\alpha = \alpha_1^\nu * \dots * \alpha_\omega^\nu$, where for each * endomorphism γ and each p in ${}^*I^+$, γ^p stands for p applications of γ , e.g., $\gamma^2 = \gamma * \gamma$. Then α is a * endomorphism of ${}^*\mathcal{Y}$.

Note that for each k in I^+ and each p_1, \dots, p_k in ${}^*I^+$ and each n_1, \dots, n_k in I , if $1 \leq p_i \leq \omega$, then (by the commutivity of elements of S and therefore *S)

$$\alpha_{p_1}^{n_1} * \dots * \alpha_{p_k}^{n_k} * \alpha$$

in a * endomorphism of ${}^*\mathcal{Y}$. Let

$$Y' = \{\alpha_{p_1}^{n_1} * \dots * \alpha_{p_k}^{n_k} * \alpha(y) \mid y \in Y, k \in I^+, n_i \in I \\ \text{and } 1 \leq p_i \leq \omega \text{ for } i = 1, \dots, k\}.$$

Then $\alpha(Y) \subseteq Y'$, and for each β in S and y in Y' , $*\beta(y)$ and $*\beta^{-1}(y)$ are in Y' .

Let β be an element of S . Since β is an endomorphism of \mathcal{Y} and is one-to-one, $*\beta$ is an $*$ endomorphism of $*\mathcal{Y}$ and is $*$ one-to-one, from which it follows that $*\beta$ is an endomorphism of $*\mathcal{Y}$ and is one-to-one. For each j in J , let $T'_j = *T_j \upharpoonright Y'$ and let $\mathcal{Y}' = \langle Y', T'_j \rangle_{j \in J}$. For each γ in S , let $\gamma' = *\gamma \upharpoonright Y'$. Then by construction, β' is one-to-one and is an endomorphism of \mathcal{Y}' . Also since by construction $\beta'^{-1}(y)$ is in Y' for all y in Y' , β' is onto Y' . Thus β' is an automorphism of Y' . Since $\alpha \upharpoonright Y$ is an isomorphic imbedding of \mathcal{Y} into \mathcal{Y}' , we may identify Y and $\alpha(Y)$ and thus consider \mathcal{Y}' as an extension of \mathcal{Y} and β' as an extension of β . \square

THEOREM 5.2. (Theorem 2.8.). *Suppose \mathcal{L} is a monotonic prescalar structure. Then \mathcal{L} has an extension $\overline{\mathcal{L}} = \langle \overline{X}, \succeq', \overline{F}, \overline{R} \rangle$ that has the following five properties:*

- (i) $\overline{\mathcal{L}}$ is Dedekind complete;
- (ii) $\overline{\mathcal{L}}$ is homogeneous, i.e., for each x, y in X there exists an automorphism α of $\overline{\mathcal{L}}$ such that $\alpha(x) = y$;
- (iii) \overline{F} satisfies monotonicity (Definition 2.10.);
- (iv) each automorphism of \mathcal{L} extends to an automorphism of $\overline{\mathcal{L}}$;
- (v) X is dense in $\langle \overline{X}, \succeq' \rangle$, i.e., for each x, y in \overline{X} such that $x \succ' y$, there exists z in X such that $x \succ' z \succ' y$.

Proof by nonstandard analysis. Let \mathcal{E} be an enlargement of \mathcal{L} . Then $*\mathcal{L} = \langle *X, *\succeq, *F, *R \rangle$ and $*\mathcal{E} = \langle *A, *\succeq, ** \rangle$. To simplify notation, \succeq will be written for $*\succeq$ and $*$ for $**$.

Recall that an element α of $*A$ is said to be *finite* if and only if for some β, γ in A , $\beta \succeq \alpha \succeq \gamma$. α in $*A$ is said to be *infinitesimal* if and only if for each positive η in A , $\eta \succ \alpha \succ \eta^{-1}$. Define the binary relation \simeq on $*A$ as follows: for each α, β in $*A$, $\alpha \simeq \beta$ if and only if $\alpha\beta^{-1}$ infinitesimal. Then it is easy to show that \simeq is an equivalence relation on $*A$.

Definition 1. An element x in *X is said to be *finite* if and only if there exist y, z in X such that $y \succ x \succ z$.

Definition 2. Define the binary relation \cong on the set of finite elements of *X as follows: for each finite x, y in *X , $x \cong y$ if and only if for some infinitesimal α, β in *A , $\alpha(x) \succeq y$ and $\beta(y) \succeq x$.

The following three lemmas are easy to show:

Lemma 1. \cong is an equivalence relation on the set of finite elements of *X .

Lemma 2. For all finite elements x, y in *X , $x \cong y$ if and only if for all infinitesimal α, β in *A , $\alpha(x) \cong \beta(y)$ if and only if for some infinitesimal α, β in *A , $\alpha(x) \cong \beta(y)$.

Lemma 3. Suppose α in *A is positive and noninfinitesimal and β in *A is infinitesimal. Then $\alpha \succ \beta$, i.e., for each x in *X , $\alpha(x) \succ \beta(x)$.

Lemma 4. Suppose x, y, u, v are finite elements of *X , $x \cong u$ and $y \cong v$. Then ${}^*F(x, y) \cong {}^*F(u, v)$.

Proof. Without loss of generality suppose ${}^*F(x, y) \succeq {}^*F(u, v)$. Let α and β be infinitesimal elements of *A such that $\alpha(u) \succ x$ and $\beta(v) \succ y$. Let $\gamma = \max \{\alpha, \beta\}$. Then $\gamma(u) \succ x$ and $\gamma(v) \succ y$. Then by * monotonicity of *F ,

$$(5.1) \quad \gamma[{}^*F(u, v)] = {}^*F[\gamma(u), \gamma(v)] \succ {}^*F(x, y) \succeq {}^*F(u, v).$$

Since γ is infinitesimal, by Lemma 2, $\gamma[{}^*F(u, v)] \cong {}^*F(u, v)$, and thus by Equation (5.1), ${}^*F(x, y) \cong {}^*F(u, v)$.

Definition 3. Let

$$\bar{X} = \{\Delta \mid \Delta \text{ is a } \cong \text{ equivalence class of finite elements of } {}^*X\}.$$

Let \succeq' be the following binary relation on \bar{X} : for each Δ, Γ in \bar{X} ,

$$\Delta \succeq' \Gamma \quad \text{iff} \quad \text{for some } x \in \Delta \text{ and } y \in \Gamma, x \succeq y.$$

The following lemma follows from applying usual nonstandard analysis techniques to $\langle X, \succeq \rangle$:

Lemma 5. The following three statements are true:

1. For each Δ, Γ in \bar{X} , $\Delta \succ' \Gamma$ if and only if for all x in Δ and all y in Γ , $x \succ y$.
2. \succeq' is a total ordering on \bar{X} .
3. $\langle \bar{X}, \succeq' \rangle$ is Dedekind complete.

Definition 4. Define the ternary relation \bar{F} on \bar{X} as follows: for each Δ, Γ, Σ in \bar{X} , $\bar{F}(\Delta, \Gamma, \Sigma)$ if and only if for some $x \in \Delta, y \in \Gamma$, and $z \in \Sigma$, $*F(x, y) \cong z$.

Lemma 6. Let Δ, Γ and Σ be elements of \mathcal{X} . Then $\bar{F}(\Delta, \Gamma, \Sigma)$ if and only if for all $x \in \Delta, y \in \Gamma$, and $z \in \Sigma$, $*F(x, y) \cong z$.

Proof. We need only show the 'only if' part of the lemma. Suppose $\bar{F}(\Delta, \Gamma, \Sigma)$. Let u, v, w be such that $u \in \Delta, v \in \Gamma, w \in \Sigma$, and $*F(u, v) \cong w$. Let x, y be arbitrary elements of $*X$ such that $x \in \Delta$ and $y \in \Gamma$. Then $x \cong u$ and $y \cong v$. Since \cong is an equivalence relation, we need only show that $*F(x, y) \cong w$. There are two cases: *Case 1:* $*F(x, y) \succeq w$. Let α, β be infinitesimal positive elements of $*A$ such that $\alpha(u) \succeq x$ and $\beta(v) \succeq y$. Let $\gamma = \max \{\alpha, \beta\}$. Then $\gamma(u) \succeq x$ and $\gamma(v) \succeq y$, and by the $*$ monotonicity of $*F$ and Lemma 2,

$$\gamma(w) \cong \gamma[*F(u, v)] = *F[\gamma(u), \gamma(v)] \succeq *F(x, y) \succeq w,$$

and thus, since by Lemma 3, $\gamma(w) \cong w$, it follows that $*F(x, y) \cong w$. *Case 2:* that $w \succ *F(x, y)$ follows by an analogous argument.

It immediately follows from Lemma 6 that for all $\Delta, \Gamma, \Sigma, \Sigma'$ in \bar{X} , if $\bar{F}(\Delta, \Gamma, \Sigma)$ and $\bar{F}(\Delta, \Gamma, \Sigma')$, then $\Sigma = \Sigma'$, i.e., that \bar{F} is a function. Thus we shall write $\bar{F}(\Delta, \Gamma) = \Sigma$ for $\bar{F}(\Delta, \Gamma, \Sigma)$.

Definition 5. For each finite α in $*A$, let $\bar{\alpha}(\Delta, \Gamma)$ be the binary relation on \bar{X} that is defined as follows: for each Δ, Γ in \bar{X} ,

$$\alpha(\Delta, \Gamma) \quad \text{iff} \quad \text{for some } x \text{ in } \Delta, \alpha(x) \in \Gamma.$$

Lemma 7. For each finite α in $*A$ and each Δ, Γ in \bar{X} ,

$$\bar{\alpha}(\Delta, \Gamma) \quad \text{iff} \quad \text{for all } x \text{ in } \Delta, \alpha(x) \in \Gamma.$$

Proof. Suppose $\bar{\alpha}(\Delta, \Gamma)$. By Definition 5, let u in Δ be such that $\alpha(u) \in \Gamma$. Let x be an arbitrary element of Δ . Then $x \cong u$, and thus by Lemma 2, let γ, δ be infinitesimal elements of $*A$ such that $\gamma(x) \succeq u$ and $\delta(u) \succeq x$. Then $\alpha * \gamma(x) \succeq \alpha(u)$ and $\alpha * \delta(u) \succeq \alpha(x)$. Using commutivity of $*$ in $*A$, it follows that $\gamma[\alpha(x)] \succeq \alpha(u)$ and $\delta[\alpha(u)] \succeq \alpha(x)$, and since $\alpha(x)$ and $\alpha(u)$ are finite, it then follows that $\alpha(x) \cong \alpha(u)$. Since $\alpha(u) \in \Gamma$, it follows that $\alpha(x) \in \Gamma$.

Lemma 8. For each finite α in $*A$ and each Δ, Γ, Γ' in \bar{X} , if $\bar{\alpha}(\Delta, \Gamma)$ iff $\bar{\alpha}(\Delta, \Gamma')$, then $\Gamma = \Gamma'$.

Proof. Immediate from Lemma 7.

From Lemma 8 it follows that $\bar{\alpha}$ is a function on \bar{X} and we shall write $\bar{\alpha}(\Delta) = \Gamma$ for $\bar{\alpha}(\Delta, \Gamma)$.

Lemma 9. For each finite α in *A , $\bar{\alpha}$ is an automorphism of $\langle \bar{X}, \succeq', \bar{F} \rangle$.

Proof. Let α be an arbitrary finite element of *A . It is easy to show that $\bar{\alpha}$ is an order isomorphism of $\langle \bar{X}, \succeq' \rangle$. Let Δ, Γ be arbitrary elements of \bar{X} . Let x be an element of Δ and y an element of Γ . Let z and Σ be such that ${}^*F(x, y) = z$ and $\bar{F}(\Delta, \Gamma) = \Sigma$. Then $z \in \Sigma$. Furthermore, $\alpha(x) \in \bar{\alpha}(\Delta)$, $\alpha(y) \in \bar{\alpha}(\Gamma)$, and $\alpha(z) \in \bar{\alpha}(\Sigma)$. Since α is a * automorphism of ${}^*\mathcal{A}$, ${}^*F[\alpha(x), \alpha(y)] = \alpha(z)$. Thus by Definitions 4 and 5, $\bar{F}[\bar{\alpha}(\Delta), \bar{\alpha}(\Gamma)] = \bar{\alpha}(\Sigma)$.

Lemma 10. For each Δ, Γ in \bar{X} there exists a finite α in *A such that $\bar{\alpha}(\Delta) = \Gamma$.

Proof. Let Δ, Γ be arbitrary elements of \bar{X} . Let x be an element of Δ and u, v be elements of Γ such that $u \succ' v$. By * automorphism density let α be an element of *A such that $u \succ \alpha(x) \succ v$. Then α is finite and $\alpha(x) \in \Gamma$. Thus $\bar{\alpha}(\Delta) = \Gamma$.

Lemma 11. \bar{F} is strictly monotonic in each variable.

Proof. We will show \bar{F} is strictly monotonic in the first variable. The case for the second variable follows by a similar argument. Suppose Δ, Δ' , and Γ are elements of \bar{X} and $\Delta \succ' \Delta'$. We need only show that $\bar{F}(\Delta, \Gamma) \succ' \bar{F}(\Delta', \Gamma)$. Let x be an element of X and Σ the element of \bar{X} such that $x \in \Sigma$. By Lemma 10, let α be a finite element of *A such that $\bar{\alpha}(\Gamma) = \Sigma$. Let $\Theta = \bar{\alpha}(\Delta)$ and $\Theta' = \bar{\alpha}(\Delta')$. Then by Lemma 9, $\Theta \succ' \Theta'$. Also by Lemma 9,

$$\bar{F}(\Delta, \Gamma) \succ' \bar{F}(\Delta', \Gamma) \quad \text{iff} \quad \bar{F}(\Theta, \Sigma) \succ' \bar{F}(\Theta', \Sigma),$$

so that it needs only to be shown that $\bar{F}(\Theta, \Sigma) \succ' \bar{F}(\Theta', \Sigma)$. Let $y \in \Theta$ and $z \in \Theta'$. Since $\Theta \succ' \Theta'$, let u, v be elements of X such that $y \succeq u \succ v \succeq z$. Then by the * monotonicity of *F ,

$${}^*F(y, x) \succeq {}^*F(u, x) \succ {}^*F(v, x) \succeq {}^*F(z, x),$$

where ${}^*F(u, x) = F(u, x)$ and ${}^*F(v, x) = F(v, x)$ are elements of X . It follows from this that $\bar{F}(\Theta, \Sigma) \succ' \bar{F}(\Theta', \Sigma)$.

For each x in X , let \bar{x} be the element Δ of \bar{X} such that $x \in \Delta$. Then the mapping of $x \rightarrow \bar{x}$ is an isomorphic imbedding of $\langle X, \succeq, F \rangle$ into $\langle \bar{X}, \succeq', \bar{F} \rangle$. Thus we may think of $\langle \bar{X}, \succeq', \bar{F} \rangle$ as an extension of $\langle X, \succeq, F \rangle$. Considered in this way, it is easy to show that X is a dense subset of $\langle \bar{X}, \succeq' \rangle$. Let \bar{R} be the ternary relation on \bar{X} defined by: for each Δ, Γ, Σ in \bar{X} ,

$$\bar{R}(\Delta, \Gamma, \Sigma) \text{ iff } \bar{F}(\Delta, \Gamma, \Sigma).$$

Let $\bar{\mathcal{L}} = \langle \bar{X}, \bar{\succ}', \bar{F}, \bar{R} \rangle$. Then by the above Lemmas Conditions (i), (iv), and (v) of the Theorem holds. By Lemmas 9 and 10, Condition (ii) holds, and by Lemma 11, Condition (iii) holds. \square

THEOREM 5.3. (*Theorem 2.9.*). *Let $\mathcal{L} = \langle X, \succ, \circ \rangle$ be a positive concatenation structure. Then the following conditions are equivalent:*

- (i) \mathcal{L} satisfies automorphism density (Definition 2.10.);
- (ii) for each x, y in X , if $x \succ y$, then for some α in A , $x \succ \alpha(y) \succ y$;
- (iii) for each n in I^+ and each x, y in X , $n(x \circ y) = (nx) \circ (ny)$.

Proof. (i) implies (ii) by the definition of automorphisms density.

Assume (ii). Then by Theorem 2.5. of Cohen and Narens (1979), \circ is a closed operation. Then by a modification of the proof of Theorem 5.2. of Cohen and Narens (1979), it follows that \mathcal{L} is isomorphically imbeddable in a Dedekind complete scalar structure $\bar{\mathcal{L}} = \langle \bar{X}, \bar{\succ}', \bar{\circ} \rangle$ that is a positive concatenation structure. By Theorem 3.1. of Cohen and Narens (1979), for each x, y in \bar{X} and each n in I^+ , $n(x \bar{\circ} y) = (nx) \bar{\circ} (ny)$. Since \mathcal{L} is a substructure of $\bar{\mathcal{L}}$, Condition (iii) follows.

(iii) implies that α_n defined by $\alpha_n(x) = nx$ is an automorphism of \mathcal{L} , and thus by Lemma 3.3. of Cohen and Narens (1979) (i) follows. \square

THEOREM 5.4. (*Theorem 2.10.*). *Let $\mathcal{L} = \langle X, \succ, \circ \rangle$ be a totally ordered positive concatenation structure that satisfies automorphism density. Then the following three statements are true:*

1. \mathcal{L} is extendable to a positive concatenation structure that is a Dedekind complete scalar structure.
2. All Dedekind completions of \mathcal{L} that are totally ordered positive concatenation structures are isomorphic.
3. If $\bar{\mathcal{L}}$ is a Dedekind completion of \mathcal{L} that is a totally ordered positive concatenation structure, then each automorphism of \mathcal{L} extends to an automorphism of $\bar{\mathcal{L}}$.

Proof. 1. Statement 1 was shown in the proof of Condition (ii) of Theorem 5.3.

2. Suppose by Statement 1 of this theorem that $\mathcal{L} = \langle \bar{X}, \succeq', \bar{O} \rangle$ is an extension of \mathcal{L} that is a positive concatenation structure and a Dedekind complete scalar structure. We will suppose that $\bar{\mathcal{L}}$ resulted in the construction used in Theorem 5.2. of Cohen and Narens (1979). By this construction, X is an order dense subset of $\langle \bar{X}, \succeq' \rangle$. Suppose $\mathcal{L}_1 = \langle X_1, \succeq_1, O_1 \rangle$ is an extension of \mathcal{L} that is a Dedekind complete positive concatenation structure. To show Statement 2 of the Theorem, we need only show that $\bar{\mathcal{L}}$ and \mathcal{L}_1 are isomorphic.

It is easy to show by using basic properties of positive concatenation structures that $\langle \bar{X}, \succeq' \rangle$ and $\langle X_1, \succeq_1 \rangle$ are without endpoints and have denumerable order dense subsets. Since $\bar{\mathcal{L}}$ and \mathcal{L}_1 are Dedekind complete it follows that $\langle \bar{X}, \succeq' \rangle$ and $\langle X_1, \succeq_1 \rangle$ are of order type θ and therefore by a classical theorem of G. Cantor are isomorphic. Thus without loss of generality, we may assume $\bar{X} = X'$ and $\succeq' = \succeq_1$, and write $\mathcal{L}_1 = \langle \bar{X}, \succeq', O_1 \rangle$. Thus to show the Theorem, we need only show $\bar{O} = O_1$. This will be done by contradiction. Suppose x, y are elements of $\bar{\mathcal{L}}$ such that $x \bar{O} y \neq x O_1 y$.

Case 1. $x \bar{O} y \succ' x O_1 y$. Since X is order dense in \bar{X} , let z, w in X be such that

$$(5.2) \quad x \bar{O} y \succ' z \succ' w \succ' x O_1 y.$$

Since $\bar{\mathcal{L}}$ satisfies automorphism density, let β be an automorphism of $\bar{\mathcal{L}}$ such that

$$z \succ' \beta(w) \succ' w.$$

Let s be an element of X such that $x \succ s$. It then follows from basic properties of automorphisms of positive concatenation structures developed in Cohen and Narens (1979) that $\beta^n(s)$ becomes arbitrarily large for large n in I^+ and thus that for some k in I^+ ,

$$\beta^{k+1}(s) \succeq x \succ \beta^k(s),$$

which by letting $u = \beta^k(s)$ yields

$$(5.3) \quad \beta(u) \succeq' x \succ' u.$$

Similarly v in X can be found so that

$$(5.4) \quad \beta(v) \succ' y \succ' v.$$

Thus by Equations (5.3) and (5.4) and the monotonicity of \bar{O} , we get

$$(5.5) \quad \beta(u) \bar{O} \beta(v) = \beta(u) \bar{O} \beta(v) \succ' x \bar{O} y \succ' z \succ' \beta(w).$$

Since β is an automorphism of \mathcal{L} , it follows from Equation (5.5) that

$$(5.6) \quad u \bar{O} v \succ w.$$

By Equations (5.2) and (5.4), $w \succ' x \bar{O}_1 y$. Thus by Equations (5.3) and (5.4) and the monotonicity of \bar{O}_1 ,

$$w \succ u \bar{O}_1 v = u \bar{O} v,$$

which contradicts Equation (5.6).

Case 2. $x \bar{O}_1 y \succ' x \bar{O} y$. Similar to Case 1.

3. We will now show Statement 3. Let $\mathcal{L} = \langle \bar{X}, \succ', \bar{O} \rangle$ be an extension of \mathcal{L} that is a Dedekind complete positive concatenation structure. Then by Statements 2 and 1 and the proof of Statement 2, X is an order dense subset of $\langle \bar{X}, \succ' \rangle$ and \mathcal{L} is a scalar structure. Let α be an arbitrary automorphism of \mathcal{L} , and for each x in \bar{X} , let

$$(5.7) \quad \bar{\alpha}(x) = \text{l.u.b. } \{ \alpha(y) \mid y \in X \text{ and } x \succ' y \}.$$

We will show $\bar{\alpha}$ is an automorphism of $\bar{\mathcal{L}}$.

The 'l.u.b.' of Equation (5.7) exists by the order density of X in $\langle \bar{X}, \succ' \rangle$. Thus $\bar{\alpha}$ is a function from \bar{X} into \bar{X} .

Suppose x, y are arbitrary elements of \bar{X} and $x \succ' y$. To show $\bar{\alpha}$ is order preserving it is sufficient to show $\bar{\alpha}(x) \succ' \bar{\alpha}(y)$. Since X is order dense in $\langle \bar{X}, \succ' \rangle$, let u, v in X be such that $x \succ' u \succ' v \succ' y$. Then it follows from Equation (5.7) that

$$\bar{\alpha}(x) \succ' \bar{\alpha}(u) \succ' \bar{\alpha}(v) \succ' \bar{\alpha}(y).$$

Next we will show that $\bar{\alpha}$ is onto \bar{X} . Let x be an arbitrary element of \bar{X} , $Y = \{y \in \bar{X} \mid \bar{\alpha}(y) \succ' x\}$ and $Z = \{y \in \bar{X} \mid x \succ' \bar{\alpha}(y)\}$. Then it easily follows that (Y, Z) is a Dedekind cut of $\langle \bar{X}, \succ' \rangle$. By Dedekind completeness of $\langle \bar{X}, \succ' \rangle$, let w be the cut element in $\langle \bar{X}, \succ' \rangle$ of (Y, Z) . We will show by contradiction that $\bar{\alpha}(w) = x$. Suppose $\bar{\alpha}(w) \neq x$. There are two cases to be considered:

Case 1. $\bar{\alpha}(w) \succ' x$. Since X is order dense in $\langle \bar{X}, \succeq' \rangle$, let p in X be such that $\bar{\alpha}(w) \succ' p \succ' x$. Since α is an automorphism of \mathcal{L} , let q in X be such that $\alpha(q) = p$. Then q is in Y and $w \succ' q$, contradicting that w is the cut element of $\langle Y, Z \rangle$.

Case 2. $x \succ' \bar{\alpha}(w)$. Similar to Case 1.

Again assume that x, y are arbitrary elements of X' . We will show $\bar{\alpha}(x \bar{\circ} y) = \bar{\alpha}(x) \bar{\circ} \bar{\alpha}(y)$. Since by Statement 2, all Dedekind complete, totally ordered positive concatenation structures that are extensions of \mathcal{L} are isomorphic, it follows from the construction used in the proof of Theorem 5.2. of Cohen and Narens (1979) that for all w, z in \bar{X} ,

$$(5.8) \quad w \bar{\circ} z = \text{l.u.b} \{u \circ v \mid u, v \in X, w \succeq' u, \text{ and } z \succeq' v\}.$$

In particular, from Equation (5.8) it follows that

$$(5.9) \quad \{p \mid p \in X \text{ and } x \bar{\circ} y \succeq' p\} \\ = \{p \mid p \in X \text{ and for some } u, v \text{ in } X, x \succeq' u, y \succeq' v, \\ \text{and } u \circ v \succeq' p\}.$$

Also, by Equation (5.7),

$$(5.10) \quad \bar{\alpha}(x) \bar{\circ} \bar{\alpha}(y) = \text{l.u.b} \{s \circ t \mid s, t \in X, \bar{\alpha}(x) \succeq' s, \text{ and } \bar{\alpha}(y) \succeq' t\}.$$

Letting u, v in X be such that in Equation (5.10) $\alpha(u) = s$ and $\alpha(v) = t$ and using $\bar{\alpha}$ is order preserving on $\langle \bar{X}, \succeq' \rangle$ and noting from Equation (5.7) $\bar{\alpha}$ is an extension of α , it follows that

$$(5.11) \quad \bar{\alpha}(x) \bar{\circ} \bar{\alpha}(y) = \text{l.u.b} \{\alpha(u) \circ \alpha(v) \mid u, v \in X, \bar{\alpha}(x) \succeq' \alpha(u) \\ \text{and } \bar{\alpha}(y) \succeq' \alpha(v)\} \\ = \text{l.u.b} \{\alpha(u) \circ \alpha(v) \mid u, v \in X, x \succeq' u \text{ and } y \succeq' v\} \\ = \text{l.u.b} \{\alpha(u \circ v) \mid u, v \in X, x \succeq' u \text{ and } y \succeq' v\}.$$

However, from Equation (5.9)

$$(5.12) \quad \text{l.u.b} \{\alpha(u \circ v) \mid u, v \in X, x \succeq' u, \text{ and } y \succeq' v\} \\ = \text{l.u.b} \{\alpha(u \circ v) \mid u, v \in X \text{ and } x \bar{\circ} y \succeq' u \bar{\circ} v\}.$$

Thus from Equations (5.11), (5.12), and (5.7),

$$\begin{aligned}\bar{\alpha}(x) \bar{\circ} \bar{\alpha}(y) &= \text{l.u.b } \{\alpha(u \circ v) \mid u, v \in X \text{ and } x \circ' y \succeq' u \circ v\} \\ &= \bar{\alpha}(x \bar{\circ} y). \quad \square\end{aligned}$$

Throughout the rest of this section, we will use concepts developed in Narens and Luce (1976) and Cohen and Narens (1979).

DEFINITION 5.1. Let $\mathcal{E} = \langle X \times P, \succ \rangle$ be a local conjoint structure with minimal element ab (Definition 4.1. of Narens and Luce, 1976). $\langle \alpha, \beta \rangle$ is said to be a *change of scale* for \mathcal{E} if and only if α and β are order automorphisms of $\langle X, \succ_X \rangle$ and $\langle P, \succ_P \rangle$ respectively, and for each xp, yq in $X \times P$,

$$xp \sim yq \quad \text{iff} \quad \alpha(x)\beta(p) \sim \alpha(y)\beta(q). \quad \square$$

DEFINITION 5.2. $\mathcal{E} = \langle X \times P, \succ, ab \rangle$ is said to be a *conjoint scalar structure* (with minimal element ab) if and only if \mathcal{E} satisfies the following four conditions:

- (i) \mathcal{E} is a local conjoint structure with minimal element ab ;
- (ii) $\langle X, \succ_X \rangle$ and $\langle P, \succ_P \rangle$ are totally ordered and Dedekind complete;
- (iii) for each xp in $X \times P$ there exists y such that $yb \sim xp$;
- (iv) for each x, y in X , if $x \neq a$ and $y \neq a$ then there exists change of scale (α, β) such that $\alpha(x) = y$. \square

(The Dedekind completeness in Condition (ii) is not essential and the homogeneity part of Condition (iv) can be weakened to an automorphism density condition.)

CONVENTION. Throughout the rest of this section let $\mathcal{E} = \langle X \times P, \succ, ab \rangle$ be a fixed conjoint scalar structure. Since \succ_X and \succ_P are total orderings, we will often write \succeq_X for \succ_X and \succeq_P for \succ_P . \square

CONVENTION. Throughout the rest of this section let $\mathcal{E}^+ = \langle X^+, \succeq', \circ' \rangle$ be the positive concatenation structure induced by \mathcal{E} (Narens-Luce 1976, Definition 4.3.).

THEOREM 5.5. \mathcal{E}^+ is a Dedekind complete, scalar structure.

Proof. Let $\langle \alpha, \beta \rangle$ be a change of scale for \mathcal{E} . We need only show that α (restricted to X^+) is an automorphism of X^+ ; that is, since α is an order automorphism of $\langle X, \succeq_X \rangle$, we need only show $\alpha(x \circ' y) = \alpha(x) \circ' \alpha(y)$ for all x, y in X^+ .

Let ξ, τ be defined as follows: for each xp in $X \times P$,

$$xp \sim \xi(xp)b \quad \text{and} \quad xb \sim a\tau(x).$$

Then by the construction of Narens-Luce (1976),

$$x \circ' y = \xi[x\tau(y)].$$

Note that since α, β are order automorphisms of $\langle X, \succeq_X \rangle$ and $\langle P, \succeq_P \rangle$ and a, b are minimal elements, it follows that

$$\alpha(a) = a \quad \text{and} \quad \beta(b) = b.$$

Let x, y be arbitrary elements of X . Then

$$\begin{aligned} yb &\sim a\tau(y), \\ \alpha(y)\beta(b) &\sim \alpha(a)\beta[\tau(y)], \\ \alpha(y)b &\sim a\beta[\tau(y)], \end{aligned}$$

and thus

$$(5.13) \quad \tau[\alpha(y)] = \beta[\tau(y)].$$

Now,

$$\begin{aligned} (x \circ' y)b &\sim x\tau(y), \\ \alpha(x \circ' y)\beta(b) &\sim \alpha(x)\beta[\tau(y)], \\ \alpha(x \circ' y)b &\sim \alpha(x)\beta[\tau(y)]. \end{aligned}$$

Thus by Equation (5.13),

$$\alpha(x \circ' y)b \sim \alpha(x)\tau[\alpha(y)],$$

from which it follows that

$$\begin{aligned} \alpha(x \circ' y)b &\sim \xi[\alpha(x)\tau[\alpha(y)]]b, \\ \text{i.e.,} \quad \alpha(x \circ' y) &= \alpha(x) \circ' \alpha(y). \quad \square \end{aligned}$$

DEFINITION 5.3. $\langle \phi, \psi, \odot \rangle$ is said to be a *scalar representation* for \mathcal{E} if and only if ϕ is a function from X onto the nonnegative reals, ψ is a function from P onto the nonnegative reals, \odot is a binary operation on the nonnegative reals, and the following five conditions hold for each x, y, q in $X \times P$,

- (i) $\phi(a) = \psi(b) = 0$;
- (ii) $\phi(x) \odot \psi(b) = \phi(x)$;
- (iii) $\phi(a) \odot \psi(p) = \psi(p)$;
- (iv) there exist $f: Re^+ \rightarrow Re^+$ such that if $x \succ_X a$ and $p \succ_P b$, then

$$\phi(x) \odot \psi(p) = \psi(p) f \left[\frac{\phi(x)}{\psi(p)} \right]$$

- (v) $xp \succsim yq$ iff $\phi(x) \odot \psi(p) \geq \phi(y) \odot \psi(q)$. \square

It follows from Condition (iv) of Definition 5.3. that multiplications by positive reals are the automorphisms of the positive concatenation structure $\langle Re^+, \geq, \odot \rangle$. It also follows from Condition (iv) that if $\langle \phi, \psi, \odot \rangle$ is a scalar representation for \mathcal{E} , then $\langle r\phi, r\psi, \odot \rangle$ is a scalar representation for \mathcal{E} for all r in Re^+ .

THEOREM 5.6. *There exists a scalar representation for \mathcal{E} .*

Proof. By Theorem 3.2. of Cohen and Narens (1979), let $\langle \phi', f \rangle$ be a unit representation for the fundamental unit structure \mathcal{E}^+ . Define ϕ as follows: $\phi(a) = 0$ and for each x in X^+ , $\phi(x) = \phi'(x)$. Define ψ on P as follows: for each p in P ,

$$\psi(p) = \phi[\xi(ap)].$$

Then $\psi(b) = \phi(a) = 0$. Define \odot as follows: for each nonnegative real r, s ,

$$r \odot s = r \quad \text{if } s = 0,$$

$$r \odot s = s \quad \text{if } r = 0,$$

$$r \odot s = s \cdot f \left[\frac{r}{s} \right] \quad \text{if } r > 0, s > 0.$$

Then by construction, we have shown Conditions (i), (ii), (iii), and (iv) by Definition 5.3.

To show Condition (v), let xp, yq be arbitrary elements of $X \times P$. Suppose $xp \succ yq$. Let $z = \xi(ap)$ and $w = \xi(aq)$. Then $\tau(z) = p$, $\tau(w) = q$, $\phi(z) = \psi(p)$, and $\phi(w) = \psi(q)$. Thus

$$\begin{aligned} x\tau(z) &\succ y\tau(w), \\ \xi[x\tau(z)] &\succ_X \xi[y\tau(w)], \\ x \circ' z &\succ_X y \circ' w, \\ \phi(x \circ' z) &> \phi(y \circ' w), \\ \phi(z) \cdot f \left[\frac{\phi(x)}{\phi(z)} \right] &> \phi(w) \cdot f \left[\frac{\phi(y)}{\phi(w)} \right], \\ \psi(p) \cdot f \left[\frac{\phi(x)}{\psi(p)} \right] &> \psi(q) \cdot f \left[\frac{\phi(y)}{\psi(q)} \right], \\ \phi(x) \odot \psi(p) &> \phi(y) \odot \psi(q). \quad \square \end{aligned}$$

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