Dedekind Completeness & Denumerability

Let $\mathfrak{X} = \langle X \preceq \rangle$ be a totally ordered set. Then $\mathfrak{X}$ is said to be **Dedekind complete** if and only if for each nonempty subset Z of X, there exists an element a in X, called the **supremuim of Z (in X)**, or $\sup Z$ for short, such that

(*i*) $z \preceq a$ for for all z in Z, and

(*ii*) if b in X is such that $z \preceq b$ for for all z in Z, then $b = a$.

A set Y is said to be **denumerable** if and only if there exists a one-to-one function f from $\mathbb{I}^+$ onto Y.

Let $\langle X, \preceq \rangle$ be a totally ordered set and f a function from a subset Y of X into X. Then f is said to be **$\preceq$-strictly increasing** if and only if for all x and y in X, if $x \prec y$ then $f(x) \prec f(y)$.

Continuum

$\langle X, \preceq \rangle$ is said to be a **continuum** if and only if the following four statements are true:

1. **Total ordering**: $\preceq$ is a total ordering on X.
2. **Unboundedness**: $\langle X, \preceq \rangle$ has no $\preceq$-greatest or $\preceq$-least element.
3. **Denumerable density**: There exists a denumerable subset Y of X such that for each x and z in X, if $x \prec z$ then there exists y in Y such that $x \prec y$ and $y \prec z$.
4. **Dedekind completeness**: $\langle X, \preceq \rangle$ Dedekind complete.

Cantor's Continuum Theorem

Theorem (Cantor 1895): *(existence)* $\mathfrak{X} = \langle X, \preceq \rangle$ is a continuum if and only if $\mathfrak{X}$ isomorphic to $\langle \mathbb{R}^+, \leq \rangle$.

Theorem: *(uniqueness)* Let $\mathfrak{X} = \langle X, \preceq \rangle$ be a continuum and $\mathcal{F}$ be the set of isomorphisms of $\mathfrak{X}$ onto $\langle \mathbb{R}^+, \leq \rangle$. Then for each f in $\mathcal{F}$
$\mathcal{F} = \{g * f \mid g$ is a strictly increasing function
from $\mathbb{R}^+$ onto $\mathbb{R}^+\}$.

Scale Families

Let Y be a nonempty set. A **scale family** (or *scale*) on Y is a nonempty set of functions from Y onto a subset of $\mathbb{R}$.

Let $\mathcal{F}$ be a scale family on Y. Then the elements of $\mathcal{F}$ are called **measuring functions.**

Some Scale Types

$\mathcal{F}$ is said to be a **ratio scale** if and only if

(1) $\mathcal{F}$ is a scale family, and

(2) for any f in $\mathcal{F}$,

$$\mathcal{F} = \{rf \mid r \in \mathbb{R}^+\}.$$

$\mathcal{F}$ is said to be an **interval scale** if and only if

(1) $\mathcal{F}$ is a scale family, and

(2) for any f in $\mathcal{F}$,

$$\mathcal{F} = \{rf + s \mid r \in \mathbb{R}^+ \ \& \ s \in \mathbb{R}\}.$$

$\mathcal{F}$ is said to be an **ordinal scale** if and only if

(1) $\mathcal{F}$ is a scale family, and

(2) for any f in $\mathcal{F}$, $\mathcal{F} = \{g * f \mid g$ is a strictly increasing function from $\mathbb{R}^+$ onto $\mathbb{R}^+\}$.

Narens & Luce (1976)

Extend the modern representational approach by replacing extensive structures with structures of the form $\langle X, \preceq, \oplus \rangle$ satisfying the same axioms as extensive structures, except possibly *associativity*. Such structures are called **PCS**s **(with a partial operation)**, or just **PCS**s when the operation is closed.

In particular, Narens & Luce showed that the major results of *Foundations of Measurement, Vol. 1* could be generalized using PCSs with partial or closed operations in place of extensive structures.

Existence & Uniqueness of Representations

$\mathfrak{X} = \langle X, \preceq, \oplus \rangle$ is a PCS.

Narens & Luce's formulation of PCSs with partial or closed operations used the following axiom: **half elements:** for each x there exists y such that $y \oplus y = x$.

With half-elements they showed that there exist a structure $\mathfrak{N} = \langle R, \leq, \oplus \rangle$, $R \subseteq \mathbb{R}^+$, such that

- **(existence)** there exists an isomorphism from $\mathfrak{X}$ onto $\mathfrak{N}$, and
- **(uniqueness)** for all isomorphisms φ and ψ from $\mathfrak{X}$ onto $\mathfrak{N}$, if for some a in X, $\varphi(a) = \psi(a)$, then $\varphi = \psi$.

Symmetries of PCSs

$\mathfrak{X} = \langle X, \preceq, \oplus \rangle$ is a PCS with a partial or closed operation. Then α is said to be a **symmetry** of $\mathfrak{X}$ if and only if α is a function from X onto itself such that for all x and y in X,

$$x \preceq y \text{ iff } \alpha(x) \preceq \alpha(y)$$

and

$$\alpha(x \oplus y) = \alpha(x) \oplus \alpha(y)\,.$$

$\mathfrak{X}$ is said to be **homogeneous** if and only if for all a and b in X, there exists a symmetry α of $\mathfrak{X}$ such that $\alpha(a) = b$.

Cohen & Narens (1979)

Cohen: Half-elements not needed. His method of proof used **symmetries**.

Narens: Homogeneous PCSs are ratio scalable.

A PCS $\mathfrak{X} = \langle X, \preceq, \oplus \rangle$ is said to be **homogeneous** if and only if for each x and y in X there exists a symmetry α of $\mathfrak{X}$ such that

$$\alpha(x) = y\,.$$

n-copy operator

$\mathfrak{X} = \langle X, \preceq, \oplus \rangle$ is a PCS.

Let $1x = x$ and for each positive integer n, let $(n+1)x = (nx) \oplus x$. nx is called the n-**copy operator** of $\mathfrak{X}$.

Theorem The following two statements are equivalent:

1. $\mathfrak{X}$ is homogeneous.

2. For each positive integer n, the n-copy operator is a symmetry of $\mathfrak{X}$.

Examples of PCSs

(1) $\langle \mathbb{R}^+, \leq, + \rangle$.

(2) $\langle \mathbb{R}^+, \leq, \oplus_1 \rangle$, where $r \oplus_1 s = r + s + r^{\frac{1}{3}} s^{\frac{2}{3}}$.

(3) $\langle \mathbb{R}^+, \leq, \oplus_2 \rangle$, where $r \oplus_1 s = r + s + r^2 s^2$.
(1) and (2) are homogeneous with multiplications by positives reals as their symmetries.

(3) has the identity as its only symmetry.

Unit Representations

Theorem Suppose $\mathfrak{X} = \langle X, \preceq, \oplus \rangle$ a homogeneous PCS, $R \subseteq \mathbb{R}^+$, and φ is an isomorphism of $\langle X, \preceq, \oplus \rangle$ onto $\langle R, \leq, \odot \rangle$. Then the following two statements hold:

1. **Unit Representation**: There exists a function on R such that for all r and s in R,

$$r \odot s = s \cdot f\left(\frac{r}{s}\right) .$$

2. $\langle X, \preceq, \oplus \rangle$ is an extensive structure if and only if $f(x) = 1 + x$.

PCSs with unit representations produce ratio scale representations and they can be used to generalize extensive measurement and its uses.

Narens (1981a)

(1) Generalized the Cohen & Narens to arbitrary qualitative structures

$$\mathfrak{X} = \langle X, R_1, \ldots, R_i \rangle$$

to produce an even more general theory of ratio scale measurement.

(2) Developed a general theory for derived measurement scales for functions of several ratio scalable variables.

Luce & Cohen (1983)

Extended Narens (1981a) to situations involving interval scalable structures, e.g., structures isomorphic to

$$\langle \mathbb{R}, \leq, \oplus \rangle, \text{ where } r \oplus s = \frac{r+s}{2}\,.$$

(If an interval scalable structure is properly measurable by φ, then all other proper measures are of the form $r\varphi + s$, where $r > 0$ and s is real.)

Luce & Narens (1985)

Synthesized, reformulated, and extended the results of the previous mentioned articles concerning PCSs and its generalizations.

Applied the theory to utility theory, and provided a measurement-theoretic formulation of rank dependent utility theory for two outcome gambles.

Luce (2000) **Utility of Gains and Losses**