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MEANINGFULNESS AND THE ERLANGER PROGRAM OF FELIX KLEIN*

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"Meaningfulness" is a term that has been used in the theory of measurement to describe the qualitative or empirical significance of quantitative statements. Measurement, of course, is that part of science that is concerned with assigning quantitative entities - usually "numbers", although sometimes "points" in a geometrical space - to qualitative or empirical entities. Presumably, one of the most important reasons for employing measurement is that the quantitative entities have well-known properties and are easy to manipulate. However, an obvious problem arises in that certain quantitative manipulations - although correct mathematically - may produce results that do not have any qualitative or empirical interpretation, and more strongly, may have no "qualitative significance" at all.

This is obviously a very ancient problem, and one that has received much attention in those times when mathematics was extended to new kinds of entities and when particular sciences employed new and different kinds of mathematical modeling procedures : In informal terms, it is the problem of whether one is just "playing mathematical games" or one is "describing or uncovering important structure". This paper will report on some of my investigations into this elusive, fundamental issue. Because of its brevity, most of the discussion will be restricted to an important, classical case - Felix Klein's famous Erlanger Program for geometry.

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Since ancient times, plane Euclidean geometry has had axiomatic *synthetic* characterizations. In such characterizations, the *primitives* (the undefined terms) supposedly corresponded to basic, intuitive geometric concepts, and the axioms to intuitively true propositions about those concepts. In more recent times, plane Euclidean geometry has also had an *analytic* characterization, which in this paper will be taken to be the algebraic characterization in terms of 2-tuples of real numbers with a concept of distance defined by the metric

$$d[(x_1,x_2),(y_1,y_2)] = [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{1/2} .$$

In this analytic characterization, geometric curves are described by graphs of equations, e.g., straight lines by graphs of linear equations.

What bothered synthetically oriented geometers of the Nineteenth Century was that in the analytic characterization there were curves describable by equations that apparently did not have synthetic geometric meaning - that is, were apparently "nongeometrical". Felix Klein, in this Erlanger Program for geometry, tried to bridge this difficulty by declaring those concepts in the analytic characterization to be Euclidean geometric if and only if they were invariant under the Euclidean motions (i.e. the group of transformations generated by rotations, reflections, and translations). Kleine's idea of identifying intrinsic concepts of a well-defined domain with invariance under a particular group of transformations is still very prevalent in mathematics and science today. Klein, however, did not justify philosophically his decision to identify "geometric" with "invariant", and to my knowledge no one since him has given a satisfactory philosophical argument as to the correctness of such an identification. This paper will look very closely at this identification and its consequences.

Nineteenth Century geometry and modern measurement theory have much in common in terms of methods and concepts. Klein's approach to "geometric" is essentially identical to many measurement theorists approach to "meaningful". To keep a host of technical issues from obscuring the main points of this paper, the example of plane Euclidean geometry will be used as the focus of discussion rather than some more complicated scientific context of measurement. In order to make clear that a measurement context is really being studied, this geometric setting will be described in terms of measurement terminology :

Let χ be a model of some synthetic axiomatization of plane Euclidean geometry. It is assumed that the set X of planar points is the domain of χ, so that in this formulation individual lines and circles are sets of points. χ is called the *synthetic model*.

Let $E^2 = Re \times Re$ (where Re denotes the real numbers), and let N be a structure with domain E^2. N is called the *analytic model.*

In measurement theory, χ is called the *qualitative structure* and N the *numerical structure.* Let's assume that N has been selected in an appropriate way, and that the synthetic axioms about χ are such that the following "existence theorem" can be proved : *There exists an isomorphism from* χ *onto* N. Let's also assume that N has been selected in such a way that the graphs of linear equations are the "lines" of N, and the graphs of circular quadratic equations are the "circles" of N. (Such an axiomatic situation is common in geometry, e.g., see Hilbert 1899). In measurement theory, the set S of isomorphisms from χ onto N is called the *scale* (based on N) for χ. Elements of S are also called *representations.* Let E be the group of Euclidean motions of E^2. Under the above assumptions the following *uniqueness theorem* for S can be shown (where $*$ denotes the operation of functional composition) : *For all representations* φ *and* ψ *in* S, (i) *there exists* α *in* E *such that* $\varphi = \alpha * \psi$, *and* (ii) $\beta * \varphi$ *is in* S *for all* β *in* E.

Let φ be an element of S, and let T be the following 4-ary relation on E^2 : For all points x,y,u,v in χ,

$$T[\varphi(x),\varphi(y),\varphi(u),\varphi(v)] \text{ iff } d[\varphi(x),\varphi(y)] \geq d[\varphi(u),\varphi(v)] .$$

Then in measurement theory T is said to be *meaningful* since it is easy to show that :

$$T[\gamma(x),\gamma(y),\gamma(u),\gamma(v)] \text{ iff } T[\psi(x),\psi(y),\psi(u),\psi(v)] \quad (1)$$

for all γ,ψ in S. Since all elements of S are isomorphisms onto N, it is easy to show that T is also geometric in the Erlanger Program sense (i.e., T is invariant under Euclidean motions). To help understand the geometric nature of T synthetically, it is useful to interpret T synthetically. This is done by looking at $R = \varphi^{-1}(T)$, which is defined by,

$$R(x,y,u,v) \text{ iff } T[\varphi(x),\varphi(y),\varphi(u),\varphi(v)] ,$$

for all x,y,u,v in the domain of χ. By (1) it follows that $R = \psi^{-1}(T)$ for all ψ in S. Since S consists of isomorphisms, it easily follows that R is invariant under the automorphisms of χ.

It similarly follows that each (analytic) geometric relation or concept based on E^2 has a corresponding (synthetic) relation or concept based on X that is invariant under the automorphisms of χ. Thus for purposes of trying to understand the Erlanger Program's concept of "geometric", it is sufficient to focus on the question, "Why should automorphism invariant relations and concepts based on X be the synthetically geometric ones ? "

In order to answer this, it is convenient to formulate the problem in a general abstract setting that includes the target setting of plane Euclidean geometry. In this abstract approach, X will denote some nonempty set of qualitative objects that has certain primitive concepts $P_1,\ldots,P_n,\ldots$ of qualitative significance. (These concepts may be elements of X , relations on X , relations of relations on X , etc.). In addition a new one place predicate M will be added. The expression "M(x)" is to be read as " x is meaningful" , and M will be called the *meaningfulness predicate*. It will be taken as an undefined term, and axioms about it will be stated. For Euclidean plane geometry discussed above, X would be the set of points of the synthetic Euclidean plane, $P_1,\ldots,P_n,\ldots$ the primitive synthetic concepts used in axiomatizing synthetic Euclidean geometry, and M(x) would be interpreted as " x is synthetically geometric".

The essential idea of trying to understand the thrust of the Erlanger Program's meaningfulness concept is to give two different but equivalent axiomatic characterizations of it. The first is essentially Klein's characterization applied to the qualitative structure χ : Let G be the group of transformations on X that leave each primitive concept P_n invariant and assume the following axiom : A concept c is meaningful (i.e., M(c)) if and only if it is invariant under G . The second, which will be given shortly, has axioms that say each primitive concept is meaningful and that concepts that are appropriately definable or constructible out of meaningful concepts are also meaningful. In the second characterization, if the "appropriateness" of the definability and constructibility concepts are defensible philosophically, then the equivalence of the two characterizations can then be seen as a philosophical justification of the concept of meaningfulness inherent in the Erlanger Program.

In order to treat invariance and definability concepts in complete generality, I will assume a variant of axiomatic set theory called ZFA or "Zermelo-Fraenkel Set Theory With Atoms ". This version of set theory differs from the usual axiomatic developments found in the literature in that it postulates the existence of a set of "urelements" or "atoms". Thus, ZFA is a theory about two types of entities, sets and atoms, the latter being nonsets. The word "entity" will be used to refer to elements of the theory - that is, to either sets or atoms. ZFA is axiomatizable in a first order language L that has a binary predicate symbol $\boldsymbol{\epsilon}$ and an individual constant symbol $\boldsymbol{A}$. Of course, in interpreting L , $\boldsymbol{\epsilon}$ will stand for the set-theoretic membership relation, $\in$, and $\boldsymbol{A}$ for the set of atoms, A . The axioms for ZFA - which include the axioms of Choice and Regularity - are slightly modified versions of the usual Zermelo-Fraenkel axioms, and they will not be presented here. The modifications are very minor and are made to accomodate the nonsets, i.e., the elements of A .

The set of atoms A in the current context is to be thought of as the qualitative domain of interest, e.g., in synthetic plane geometry the set of planar points. Relations on A are given their usual set-theoretic interpretations as sets of ordered n-tuples, where an ordered 2-tuple (a,b) is by definition the set $\{\{a\},\{a,b\}\}$, etc. Thus, in this theory relations on A, relations of relations on A, and for that matter all concepts based on A are either elements of A or sets ultimately based on A and its elements. This set-theoretic way of viewing relations and higher-order concepts based on A has some advantages when considering the effects of transformations.

Let f be a transformation on A (i.e., a one-to-one function from A onto A). Suppose b is a set and f(c) has been defined for all c in b. Then, by definition,

$$f(b) = \{f(c) \mid c \in b\} . \quad (2)$$

By use of Equation 2 and transfinite induction it is easy to show that f(d) is defined for each entity d and that for all entities x and y,

$$x \in y \text{ if and only if } f(x) \in f(y) . \quad (3)$$

Let H be a set of transformations on A, and let x be an entity. By definition, x is said to be *invariant under* H, in symbols, $I_H(x)$, if and only if $f(x) = x$ for all f in H. It is easy to show that this concept of invariance coincides with the usual concept of invariance of relations, namely, for an n-ary relation R,

$I_H(R)$ if and only if for all entities $x_1,\ldots,x_n$ and all f in H,

$$R(x_1,\ldots,x_n) \text{ iff } R[f(x_1),\ldots,f(x_n)] .$$

In using ZFA as part of a formal description, the set of atoms, A, will correspond to the domain of the qualitative or empirical structure, e.g., in space-time relativity to the set of space-time points. There are some concepts of ZFA that do not depend on A, e.g., the empty set, $\emptyset$. $\emptyset$ can be viewed as the same entity no matter whether A was chosen to be the set of space-time points or the set of masses. $\emptyset$ is a *logical* entity and is definable in terms of logical concepts (i.e., concepts not dependent on A). Similarly, $\{\emptyset\}$ can be viewed as a logical entity. $\emptyset$ and $\{\emptyset\}$ are examples of "pure sets", which are defined within ZFA by the following transfinite induction :

Let $P_0 = \emptyset$.

For each ordinal α, let $P_{\alpha+1} = \mathcal{P}(P_\alpha) \cup P_\alpha$, where $\mathcal{P}$ is the power set operator.

For each nonzero limit ordinal γ, let $P_\gamma = \bigcup_{\beta<\gamma} P_\beta$

Then an entity d is said to be a *pure set* if and only if $d \in P_\delta$ for some ordinal δ.

By use of transfinite induction, it is easy to show that

$$f(d) = d \tag{4}$$

for each pure set d and each transformation f on a .

The identification of invariance under automorphisms with (qualitative) meaningfulness is captured in the following axiomatization (which along with the following axioms and theorems assume implicitly axiom system ZFA)

AXIOM SYSTEM *TM* . There exists an entity G that is a group of transformations on A and such that for all entities x ,

$$I_G(x) \text{ iff } M(x) .$$

Assume axiom system *TM*. Let G be an entity that is a group of transformations on A and such that for all entities x,

$$I_G(x) \text{ iff } M(x) .$$

Then the following five axioms are consequences of axiom system *TM*:

1. Axiom MC' *(Meaningful Comprehension')* : For all sets a and all entities $a_1,\ldots,a_n$, if $\Phi(x,u_1,\ldots,u_n)$ is a formula of *L* and $M(a_1),\ldots,M(a_n)$ and

$$a = \{x \mid \Phi(x,a_1,\ldots,a_n)\} ,$$

then M(a) .

The proof that *TM* implies MC' is somewhat complicated and will not be given here.

MC' says that any set definable in terms of meaningful entities *via* the axiom of Comprehension of ZFA is itself meaningful. Note that the meaningfulness of atoms cannot directly be established through the use of MC' , since applications of the axiom of Comprehension of ZFA yield only sets. To obtain meaningfulness of atoms, another axiom - which is a very easy consequence of axiom system *TM* - is used :

2. Axiom AL *(Atomic Legacy)* : For all atoms, if M({a}) then M(a).

In axiom system *TM* , the meaningfulness predicate M is also definable through *L* and meaningful entities : Since $I_G(G)$ holds, M(G) is true, and thus M is definable by the following :

$$M(x) \text{ iff } \varphi(G,x) ,$$

where $\varphi(y,x)$ is the formula of *L* that says y is a set of transformations on A and each element of y leaves x invariant. The definability of M in this case is an instance of a more general axiom :

3. Axiom DM* *(Definable Meaningfulness*)* : There exist a formula $\Phi(x,u_1,\ldots,u_n)$ of *L* and entities $a_1,\ldots,a_n$ such that $M(a_1),\ldots,M(a_n)$, and for all entities x ,

$$M(x) \text{ iff } \Phi(x,a_1,\ldots,a_n) .$$

From Equation 4 it immediately follows that $I_G(d)$ for each pure set d , and thus by axiom system *TM* that $M(d)$. Thus the following axiom is an immediate consequence of axiom system *TM* .

4. *Axiom* MP *(Meaningful Pure Sets)* : Each pure set is meaningful.

By Equation 2, the following axiom is an immediate consequence of axiom system *TM* :

5. *Axiom* MI *(Meaningful Inheritability)* : For all sets b if $M(c)$ for all elements c of b , then $M(b)$.

THEOREM 1 (Narens, 1988). *Axiom system TM is true if and only if axioms* MC' , AL , DM* , MP , *and* MI *are true.*

Theorem 1 shows that transformational invariance is logically equivalent to a set of axioms that for the most part allows one to "define" or "construct" new meaningful entities out of already known ones. Of these axioms, Axiom MI is most suspect as a valid definability or constructibility principle, since (in its formulation above) the set b is not specified by either formula or rule. In practice axiom MI is a powerful principle. For example, it is easy to show through transfinite induction that MI and $M(\emptyset)$ imply axiom MP .

By Theorem 1, any weakening of the conjunction of the above axioms will constitute a *generalization* of axiom system *TM*. This provides enormous flexibility in generalizing the Erlanger Program's concept of meaningfulness. This extra flexibility is important, since the Erlanger concept of meaningfulness fails to provide an adequate theory of meaningfulness in many important situations. This is especially so when the qualitative structure has the identity as its only automorphism. In such cases, the Erlanger Program will yield all concepts meaningful, which is usually highly undesirable. (Historically, it was an example of a structure with only the trivial automorphism that led to the demise of the Erlanger Program as the arbitrator of things geometric : In 1916, Albert Einstein presented his famous general theory of relativity, in which physical space-time - a situation whose geometric character could not be denied - had only the trivial automorphism).

A different - but related - meaningfulness - like issue of great mathematical importance is the status of Lebesgue nonmeasurable sets of points. Historically, this issue - which is intertwined with the status of the axiom of Choice - generated some long lasting controversies :

At the beginning of the Twentieth Century, many prominent mathematicians voiced concerns about the "reality" of Lebesgue nonmeasurable sets of points. These concerns mainly had to do with the introduction of the highly infinitistic set-theoretic techniques of Georg Cantor into analysis, especially uses of the axiom of Choice. The latter was the subject of a famous series of correspondences between the mathematicians Lebesgue, Borel, Baire and Hadamard that was published in *Bulletin de la Société Mathématique de France*, 1905. Much of the discussion in these correspondences can be looked at as an informal attempt to try and establish a meaningfulness criterion for those concepts in analysis that had a more solid and direct mathematical existence from those that only existed through some highly infinitistic application, like the axiom of Choice. (The link of these early discussions about the axiom of Choice to standard meaningfulness issues like "geometric" becomes more apparent when one considers the early literature about the counter-intuitiveness of Vitali's 1905 result about the existence of Lebesgue non-measurable sets, Hausdorff's 1914 paradox, Banach's and Tarski's 1924 paradox, and Von Neumann's 1929 general result about such paradoxes). Narens (1988) shows how Lebesgue measurability nicely fits into a axiomatic meaningfulness scheme based on axioms 1 to 4 above :

Let A be the set of Euclidean planar points. Let's identify meaningfulness and Lebesgue measurability of subsets of A . Then {a} is meaningful for each element a in A, and the pure set ∅ is meaningful. From this, it follows by transfinite induction that if axiom MI were true, then each entity would be meaningful. (Also note that if the Lebesgue measurable subsets of A were taken as primitives, then the resulting structure would only have the trivial automorphism.) Narens (1988) shows that the predicate M can be defined so that *(i)* axioms MC' , AL , DM* , and MP are true; *(ii)* axiom MI is false; *(iii)* each Lebesgue measurable subset of A is meaningful; and *(iv)* each Lebesgue nonmeasurable subset of A is not meaningful. Thus in particular it follows from this result that any subset of A that is definable from elements of A , Lebesgue measurable subsets of A , and pure sets through a formula of *L* is meaningful and therefore Lebesgue measurable.

The above result shows that axiom MI is not derivable from the conjunction of axioms MC' , AL , DM* , and MP . This is one of the independence results contained in the following theorem :

THEOREM 2 (Narens, 1988). *The following three statements are true :*

1. *The conjunction of* MC' , AL , DM* , *and* MP *does not imply* MI .
2. *The conjunction of* MC' , AL , *and* DM* *does not imply* MP .
3. *The conjunction of* MC' , AL , MP , *and* MI *does not imply* DM* .

A consequence of Theorem 2 is that the conjunction of axioms MC' and AL does not imply axioms MI , MP or DM* . These later three axioms appear to me to be very difficult to justify as valid meaningfulness principles :

The nonconstructive nature of axiom MI has already been discussed above, and because of it I believe that MI should not be taken as a necessary meaningfulness principle.

Axiom MP allows the use of all possible logical concepts in defining new meaningful entities from priorly established ones through an application of axiom MC' . These logical concepts include highly infinitistic ones, and, in particular, ones that result from applications of the axiom of Choice to pure sets. The philosophical doctrine of logicism, which was developed by G. Frege and became the central idea behind Whitehead's and Russell's famous *Principia Mathematica* , holds that pure mathematics consists of exactly the logical concepts expressible in terms of pure sets and the set-theoretic $\in$-relation. While for mathematical uses of the meaningfulness concept (such as specifying the geometric entities of Euclidean plane geometry) axiom MP may be a somewhat defensible meaningfulness principle, it is far less so for scientific applications, since it is much more difficult in scientific contexts to justify highly infinitistic concepts as having any qualitative or empirical relevance. In other words, the highly platonic metaphysical assumptions inherent in logicism is incompatible with the kind of metaphysical assumptions generally made in scientific applications, and this incompatibility makes it imperative that the meaningfulness concept for the scientific applications keeps metaphysically incompatible mathematical concepts from having (meaningful) scientific import. Thus for most scientific applications, axiom MP should either be greatly weakened or eliminated.

Axiom DM* , which says that the meaningfulness predicate M is definable through a formula of set theory and meaningful entities, is difficult to justify. It was included in the axioms above since it is a consequence of Erlanger Program that is independent of the other axioms (Theorem 2). While axiom DM* is a highly desirable meaningfulness property, I see no reason to include it as a necessary condition for meaningfulness.

Axioms MC' and AL appear to me to be much more reasonable meaningfulness principles, except that axiom MC' appears to be in some ways too powerful :

Axiom MC' says that a set that is definable through a formula of and meaningful entities is itself meaningful. The problem with this principle is that rather abstract - in fact, infinitely abstract - sets can be defined in ZFA through formulas of L and the set A . Thus in scientific meaningfulness contexts, MC' would allow the meaningfulness of some infinitely abstract objects. From many philosophical perspectives this is highly undesirable, and for this (and other reasons) it seems reasonable that axiom MC' should be weakened.

Axiom AL is a perfectly straightforward and reasonable meaningfulness condition.

Thus to summarize, the concept of meaningfulness inherent in Felix Klein's Erlanger Program can be formalized as axiom system $\mathcal{TM}$. It can be shown (Theorem 1) that this axiom system is logically equivalent of the conjunction of five meaningfulness principles - MC' , AL , DM* , MP , and MI - that stress meaningfulness as a definability concept. In terms of these five principles, the philosophical commitments inherent in Erlanger Program meaningfulness concept become somewhat clarified, and it appears that they are inadequate for scientific applications, since they embrace unacceptably strong, infinitistic methods. A weakened version of these principles which assumes AL and weakened forms of MC' and MP appears to be a realistic avenue for developing a more robust and philosophically sound meaningfulness concept. (This and other methods of weakening the assumptions of Erlanger Program are discussed in detail in Narens, 1988).

One of the main applications of the Erlanger Program's meaningfulness concept has been to rule out nonmeaningful entities from consideration. This practice can be intuitively justified by Theorem 1 as follows :

Suppose in a particular setting we are interested in finding the functional relationship of the qualitative variables x , y , and z . We believe that the primitive relations (which are known) completely characterize the current situation. Furthermore, our understanding (or insight) about the situation tells us that x must be a function of y and z . (This is the typical case for an application of dimensional analysis in physics). This unknown function - which we will call "the desired function" - must be determined by the primitive relations and the qualitative variables x , y and z . Therefore, it should somehow be "definable" from these relations and variables. Even though the exact nature of the definability condition is not known, (it can be argued that) it must be weaker than the enormously powerful methods of definability encompassed by the conjunction of axioms MC' , AL , DM* , MP , and MI . Thus by Theorem 1 we know that any function relating the variable x to the variables y and z that is *not* invariant under the automorphisms of the primitives cannot be the desired function. In many situations, this knowledge *of knowing that functions not invariant under the automorphisms of the primitives cannot be the desired function* can be used to effectively find or narrow down the possibilities for the desired function.

It appears likely to the author that the near future will bring better theories of meaningfulness that will more precisely specify the nature of definability properties of the meaningfulness predicate, and that this additional knowledge will likely prove useful in strengthening the techniques of dimensional analysis of physics and other meaningfulness methods of drawing inferences about qualitative relationships.

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