On the Scales of Measurement

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Let $\mathscr{X} = \langle X, \geqslant, R_1, R_2,... \rangle$ be a relational structure, $\langle X, \geqslant \rangle$ be a Dedekind complete, totally ordered set, and n be a nonnegative integer. $\mathscr{X}$ is said to satisfy n-point homogeneity if and only if for each $x_1,..., x_n, y_1,..., y_n$ such that $x_1 > x_2 > \cdots > x_n$ and $y_1 > y_2 > \cdots > y_n$, there exists an automorphism α of $\mathscr{X}$ such that $\alpha(x_i) = y_i$. $\mathscr{X}$ is said to satisfy n-point uniqueness if and only if for all automorphisms β and γ of $\mathscr{X}$, if β and γ agree at n distinct points of $\mathscr{X}$, then β and γ are identical. It is shown that if $\mathscr{X}$ satisfies n-point homogeneity and n-point uniqueness, then $n \leqslant 2$, and for the case $n = 1$, $\mathscr{X}$ is ratio scalable, and for the case $n = 2$, interval scalable. This result is very general and may in part provide an explanation of why so few scale types have arisen in science. The cases of 0-point homogeneity and infinite point homogeneity are also discussed.

PART 1: CONCEPTS AND THEOREMS

Introduction

Almost all sciences use numbers. These numbers appear throughout all levels of the complex chain of mathematical, logical, and heuristic analyses that constitute scientific explanation and argumentation. Usually the first place they appear is in the quantification of empirical concepts. This is obviously a very important step in scientific formulation, but one for which there exists surprisingly little research given its obvious foundational and universal character. This step is usually called *measurement.*

The measurement of an empirical variable is a consistent assignment of numbers to the variable. Such an assignment is called a *scale* for the variable (Stevens, 1951). A variable may have several scales, and how these scales relate to one another determines the *scale type* of the measurement process. Many types of scales have arisen in science, and by far the most important of these are the *ordered* scales—ones for which there is a natural ordering of the empirical variable, which under measurement maps into the numerical $\geqslant$ relation of the real number system. So far in science only three types of ordered scales are in wide use, and while other types of ordered scales have been suggested and occasionally used, their impact on science has been very minimal. The three widely used types usually go by the names "ratio,"

"interval," and "ordinal." *Ratio scales* are ones for which the proper assignments of numbers to the empirical variable consist of positive real multiples of any single proper assignment; *interval* are ones for which the proper assignments consist of linear transformations $r\varphi + s$ of any single proper assignment φ, where r ranges over the positive reals and s over the reals; and *ordinal* are ones for which the proper assignments consist of all strictly monotonic transformations of any single proper assignment where the resulting transformation has the same range as the given assignment.

Obviously if one assumes that a particular type of scale is proper for the measurement of a variable, then one is implicitly making "empirical" assumptions about the variable that allows for the existence of the scale type. Just what these empirical assumptions are is not at all obvious. For example, in the case of ratio scalability, it is not at all obvious as to what corresponds empirically to the highly structured, abstract concept of multiplication by a positive real. Until very recently (Narens, 1980), this kind of issue was sidestepped by the literature, which concerned itself instead with the derivation of scale types from highly specific properties of empirical models. This approach taken in the literature does not clarify questions about the possible range of scale types and does not lead to a rational classification of scale types.

In this paper a very general approach to ordered scale types is given. Concepts and theorems are developed which allow for a reasonable classification of scale types, and these are used to show that the traditional ones—ratio, interval, and ordinal—share a core of properties that are not readily generalizable to other kinds of ordered scale types.

The paper is divided into two parts: Part 1 develops concepts and states theorems; Part 2 gives proofs of theorems in Part 1 that are not referenced to other papers.

Dedekind Completeness

Throughout this paper, we will concentrate on ordered structures whose measurements are onto the reals or positive reals. This obviously imposes certain structural and topological conditions on the empirical ordering relation, and these assumptions are specified in the following definition:

DEFINITION 1.1. $\langle X, \succcurlyeq \rangle$ is said to be of *order type* θ if and only if $\succcurlyeq$ is a binary relation on the nonempty set X and the following four conditions hold:

(1) *Total ordering.* $\succcurlyeq$ is a total ordering.

(2) *No endpoints.* For each x in X there exists y, z in X such that $y \succ x \succ z$.

(3) *Denumerable density.* There exists a denumerable subset Y of X such that for each x, z in X, if $x \succ z$, then for some y in Y, $x \succ y \succ z$.

(4) *Dedekind completeness.* For each nonempty subset Y of X, if there exists an upper bound of Y (i.e., there exists x in X such that $x \succcurlyeq y$ for all y in Y), then

there exists a least upper bound (l.u.b) for Y (i.e., there exists an upper bound z of Y such that for all upper bounds u of Y, $u \geqslant z$). ∎

G. Cantor (1895) showed the following theorem:

THEOREM 1.1.   *Suppose* $\langle X, \geqslant \rangle$ *is of order type* θ. *Then there exist functions* f *and* g *from* X *onto the reals and positive reals respectively such that for all* x, y *in* X,

$$x \geqslant y \qquad iff \qquad f(x) \geqslant f(y) \qquad iff \qquad g(x) \geqslant g(y). \quad ∎$$

There are measurement situations where the empirical ordering is not of order type θ. In some of these, the ordering relation is not a total order. However, in many of these situations (e.g., weak orders, semiorders), there are naturally induced orderings which are total, and for measurement purposes these can serve as an empirical orderings for the establishment of scales. There are of course ordered structures that have natural endpoints, and these pose no problem for the development presented here; they are excluded only as a matter of convenience to simplify definitions and statements of theorems. Denumerable density is a more serious concern. It can be argued that empirical structures are really finite and thus infinite structures should not be used for describing empirical phenomena. However, much of science is concerned with large finite structures, and it can be argued that certain infinite structures are "idealizations" of large finite ones, and what we are really dealing with in measurement and calling "empirical" are these kinds of idealizations. Such a view is implicit in the thinking and procedures of most scientists, but to my knowledge has not been rigorously justified by theorems which show that the use of these infinite idealizations in science are accurate approximations of the corresponding proper quantitative analyses for sufficiently large finite structures. Dedekind completeness presents a different kind of problem. In the presence of the other conditions for order type θ, Dedekind completeness implies that the structure is infinite and non-denumerable. While one might be able to accept certain denumerable models as idealizations of empirical settings, it is much harder to accept infinite, non-denumerable models as idealizations. However, in many measurement situations, a denumerable structure can be imbedded in a Dedekind complete one in such a way that the Dedekind complete structure inherits the denumerable structure's measurement-theoretic properties. This is discussed in some detail in Narens (1980).

Isomorphic Representations

DEFINITION 1.2.   Let $\mathscr{X} = \langle X, R_0, R_1,... \rangle$ be a *relational structure* (i.e., let X—the *domain of discourse* of $\mathscr{X}$—be a nonempty set and R_0, R_1,...—the *primitives* of $\mathscr{X}$—be relations on X), Re be the set of real numbers, Re^+ the set of positive real numbers, and I^+ the set of positive integers. φ is said to be an $\mathscr{N}$-*representation* for $\mathscr{X}$ if and only if $\mathscr{N}$ is a relational structure with domain of discourse a subset of Re and φ is a *homomorphism* of $\mathscr{X}$ into $\mathscr{N}$, i.e., $\mathscr{N} = \langle N, S_0, S_1,... \rangle$, where $N \subseteq Re$, and

S_i is an n_i-ary relation on N, where n_i is such that R_i is an n_i-ary relation, and for each i in I^+ and each $x_1,...,x_{n_i}$ in X,

$$R_i(x_1,...,x_{n_i}) \qquad \text{iff} \quad S_i[\varphi(x_1),...,\varphi(x_{n_i})].$$

φ is said to be a *representation onto* $\mathcal{N}$ if and only if φ is an $\mathcal{N}$-representation and φ is onto the domain of discourse of $\mathcal{N}$. φ is said to be an *isomorphic* $\mathcal{N}$-*representation* for $\mathcal{X}$ (or equivalently, an *isomorphism from* $\mathcal{X}$ *onto* $\mathcal{N}$) if and only if φ is a one-to-one function and φ is an $\mathcal{N}$-representation for $\mathcal{X}$ that is onto $\mathcal{N}$.

Suppose $\mathcal{X}$ is a relational structure with domain of discourse X, φ is an isomorphic $\mathcal{N}$-representation for $\mathcal{X}$, and $\mathcal{N}$ has domain of discourse N. Then for each subset Y of X, each n-ary relation R on X, and each set H of relations on X, $\varphi(Y) = \{\varphi(y) \mid y \in Y\}$, $\varphi(R)$ is the n-ary relation S on N such that for each $u_1,...,u_n$ in N,

$$S(u_1,...,u_n) \qquad \text{iff} \quad R[\varphi^{-1}(u_1),...,\varphi^{-1}(u_n)],$$

and $\varphi(H) = \{\varphi(T) \mid T \in H\}$. $\varphi(Y)$, $\varphi(R)$, and $\varphi(H)$ are called the *images of* Y, R, and H respectively *under* φ. ∎

Note that if $\mathcal{X} = \langle X, \succcurlyeq, R_1, R_2,... \rangle$, where $\succcurlyeq$ is a total ordering on $\mathcal{X}$ and $\mathcal{N} = \langle N, \geqslant, S_1, S_2,... \rangle$, where $N \subseteq Re$, then it easily follows that each $\mathcal{N}$-representation of $\mathcal{X}$ is a one-to-one function.

Let $\mathcal{X} = \langle X, \succcurlyeq, R_1, R_2,... \rangle$, where $\succcurlyeq$ is a total ordering on X. Traditionally, the theory of measurement for $\mathcal{X}$ has proceeded by specifying a numerical structure $\mathcal{N} = \langle N, \geqslant, S_1, S_2,... \rangle$, where $N \subseteq Re$, and considering $\mathcal{N}$-representations of $\mathcal{X}$. If $\mathcal{N}$ is isomorphic to $\mathcal{X}$, then this procedure is easy to justify since by isomorphism all scales are "perfect" correspondences between $\mathcal{X}$ and $\mathcal{N}$. However, if some $\mathcal{N}$-representations are not onto, then it is much more difficult to justify the measurement process, and to my knowledge there is in the literature no serious attempt to do so. To see some of the problems with non-onto representations, consider the following example:

Let f and g be the functions on Re^+ defined by

$$f(x) = 1 - \frac{1}{1+x} \qquad \text{and} \qquad g(x) = 2 - \frac{1}{1+x^2},$$

and let O_1 and O_2 be binary operations defined on the open intervals $(0,1)$ and $(1,2)$, respectively, by

$$x \, O_1 \, y = f[f^{-1}(x) + f^{-1}(y)]$$

and

$$u \, O_2 \, v = g[g^{-1}(u) + g^{-1}(v)],$$

and let O be a binary operation on Re^+ that is an extension of both O_1 and O_2. Let

$\mathscr{X} = \langle Re^+, \geqslant, + \rangle$ and $\mathscr{N} = \langle Re^+, \geqslant, \bigcirc \rangle$. (Note that in "$\mathscr{X} = \langle Re^+, \geqslant, + \rangle$" we consider "$\geqslant$" and "$+$" to be the restrictions of $\geqslant$ and $+$ to Re^+ thus making $\mathscr{X}$ a relational structure. Throughout this paper, we will observe the mathematical convention of writing obviously intended restrictions of relations and operations as the relations and operations themselves.) It is not difficult to verify that f and g are $\mathscr{N}$-representations for $\mathscr{X}$. From a measurement standpoint, the structure $\mathscr{N}$ is rather bizarre, and the representations f and g are at best capriciously related to each other, and certainly not intrinsically related through the structure $\mathscr{X}$. It seems to me that one does not want to consider $\mathscr{N}$ as a candidate for a numerical representing structure for $\mathscr{X}$, and one way to eliminate its candidacy is to demand that only onto representations be considered. Now this demand of focusing on only onto representations eliminates certain special situations where successful measurement can take place through non-onto representations, e.g., scaling the structure $\mathscr{Y} = \langle (0, 1), \geqslant, + \rangle$ into $\mathscr{M} = \langle Re^+, \geqslant, + \rangle$. In this case, the scaling is usually done so that all $\mathscr{M}$-representations of $\mathscr{Y}$ are of the form $\varphi(x) = rx$, where r is an arbitrary positive real. However, my view is that this situation should be considered as a special case, and the reason why the measurement process works here is that $\mathscr{Y}$ has only one extension with "consistent" measurement-theoretic properties, namely, $\langle Re^+, \geqslant, + \rangle$, and this extension is isomorphic to $\mathscr{M}$ and has only $\mathscr{M}$-isomorphic representations.

This paper will focus on representations that are onto. Not all measurement situations can be handled by this kind of representation (e.g., see the discussion following Example 1.3), and a more general definition of "representation" is required. One such is given in Definition 1.12 which includes $\mathscr{N}$-representations that are into as a special case.

Automorphisms

DEFINITION 1.3. Let $\mathscr{X} = \langle X, R_0, R_1, ... \rangle$ be a relational structure. α is said to be an *automorphism* of $\mathscr{X}$ if and only if α is a function from X onto X such that for $i = 0, 1, 2, ...$ and each $x_1, ..., x_{n_i}$ in X

$$R_i(x_1, ..., x_{n_i}) \quad \text{iff} \quad R_i[\alpha(x_1), ..., a(x_{n_i})].$$

Throughout this paper ι will denote the identity function on X, i.e., $\iota(x) = x$ for all x in X. It is immediate that ι is an automorphism of $\mathscr{X}$. Thus ι is often called the *identity* automorphism of $\mathscr{X}$. For all automorphisms α and β of $\mathscr{X}$, let $\alpha * \beta$ be the function on X defined by $\alpha * \beta(x) = \alpha[\beta(x)]$ for all x in X. Then it is also immediate that $\alpha * \beta$ is an automorphism of $\mathscr{X}$ for all automorphisms α, β of $\mathscr{X}$. ∎

Suppose $\mathscr{X} = \langle X, \geqslant, R_1, R_2, ... \rangle$ is a relational structure, $\geqslant$ is a total ordering on X, and φ is an isomorphic $\mathscr{N}$-representation for $\mathscr{X}$. Then it is easy to show that for each isomorphic $\mathscr{N}$-representation ψ of $\mathscr{X}$, $\varphi^{-1}\psi$ is an automorphism of $\mathscr{X}$, and for each automorphism α of $\mathscr{X}$, $\varphi\alpha$ is an isomorphic $\mathscr{N}$-representation for $\mathscr{X}$. Thus in a very natural way automorphisms and isomorphic $\mathscr{N}$-representations of $\mathscr{X}$ correspond, and

thus properties about the set of isomorphic $\mathcal{N}$-representations of $\mathcal{X}$ can be translated exactly into properties about the set of automorphisms of $\mathcal{X}$. Now this type of translation is particularly useful since the automorphisms of $\mathcal{X}$ under the operation $*$ form a rich type of mathematical structure known as a *group*, and there exists an abundance of mathematical results about groups, many of which through the above "translation" are applicable to measurement theory. Thus the tack that will be taken in this paper will be to describe the measurement-theoretic properties of $\mathcal{X}$ in terms of its automorphisms. (I prefer this to the equivalent one that uses representations of $\mathcal{X}$ since the group structure of automorphisms play the critical role in the mathematical proofs.)

Homogeneity and Uniqueness

Measurement structures can be classified in terms of properties of their representations, which by the above discussion is equivalent to a classification in terms of automorphisms. The following definition gives properties of automorphisms that will play a central role in the classification of measurement structures.

DEFINITION 1.4. Let $\mathcal{X} = \langle X, \succcurlyeq, R_1, R_2, ... \rangle$ be a relational structure, $\succcurlyeq$ a total ordering on X, H a nonempty set of automorphisms of $\mathcal{X}$, and n a nonnegative integer. Then H is said to satisfy *n-point homogeneity* if and only if for each $x_1, ..., x_n$, $y_1, ..., y_n$ in X with $x_1 > x_2 > \cdots > x_n$ and $y_1 > y_2 > \cdots > y_n$, there exists an automorphism α in H such that $\alpha(x_i) = y_i$ for $i = 1, ..., n$. (This condition is assumed to hold vacuously for $n = 0$, and thus all sets of automorphisms satisfy 0-point homogeneity.) H is said to satisfy *n-point uniqueness* if and only if for each α and β in H, if α and β agree at least at n distinct elements of X, then $\alpha = \beta$. (Thus in particular, if H satisfies 0-point uniqueness, then H consists of exactly one automorphism.) $\mathcal{X}$ is said to satisfy *n-point homogeneity* if and only if the set of automorphisms of $\mathcal{X}$ satisfies n-point homogeneity; and $\mathcal{X}$ is said to satisfy *n-point uniqueness* if and only if the set of automorphisms of $\mathcal{X}$ satisfies n-point uniqueness. ∎

Other concepts of homogeneity and uniqueness will be considered later.

It immediately follows from Definition 1.4 that $n + 1$-point homogeneity implies n-point homogeneity, and that for infinite X, n-point uniqueness implies $n + 1$-point uniqueness. It is also easy to show that if a structure satisfies n-point homogeneity then it cannot satisfy m-point uniqueness for $0 \leqslant m < n$.

In this paper, homogeneity and uniqueness are directly postulated since we consider the completely general case of relational structures. However, for many measurement applications, homogeneity and uniqueness can be deduced and/or stated in terms of particular relational properties. This is particularly true of uniqueness which is often easily derived from monotonicity or solvability assumptions. Homogeneity appears to be a trickier concept to capture. In some situations, particular automorphisms can be defined in terms of the primitives of the structure, and these automorphisms can then be used to establish homogeneity. However, there

are many measurement situations in which only the identity automorphism is definable in terms of the primitives, and in these situations homogeneity must be established by some less direct means. (These issues involving homogeneity are discussed in Narens, 1980.)

We will proceed with the classification of measurement structures by first considering cases where homogeneity and uniqueness match each other, i.e., by considering structures that simultaneously satisfy n-point homogeneity and n-point uniqueness for some n.

Absolute Structures

DEFINITION 1.5. $\mathscr{X} = \langle X, \geqslant, R_1, R_2,... \rangle$ is said to be an *absolute structure* if and only if the following three conditions are satisfied:

(1) $\mathscr{X}$ is a relational structure.

(2) $\langle X, \geqslant \rangle$ is of order type θ.

(3) $\mathscr{X}$ satisfies 0-point uniqueness.   ∎

Suppose $\mathscr{X} = \langle X, \geqslant, R_1, R_2,... \rangle$ is an absolute structure. The 0-point uniqueness of $\mathscr{X}$ implies that the identity is the only automorphism of $\mathscr{X}$ and thus in particular that $\mathscr{X}$ satisfies 0-point homogeneity. Therefore the measurement representations of $\mathscr{X}$ are very limited, consisting of at most one isomorphic representation for each potential numerical representing structure. Isomorphic representations for $\mathscr{X}$ exist: By Theorem 1.1, let φ be an isomorphism of $\langle X, \geqslant \rangle$ onto $\langle Re^+, \geqslant \rangle$, and $S_1, S_2,...$ be the respective images under φ of $R_1, R_2,...$, and let $\mathscr{N} = \langle Re^+, \geqslant, S_1, S_2 \cdots \rangle$. Then φ is an $\mathscr{N}$-representation for $\mathscr{X}$.

The following is an example of an absolute structure: Let $\oplus$ be the binary operation on Re^+ defined by: for each x, y in Re^+, $x \oplus y = x + y + x^2 y^2$. Then by Example 4.2 of Cohen and Narens (1979), $\langle Re^+, \geqslant, \oplus \rangle$ has the identity as its only automorphism and is therefore an absolute structure.

Absolute structures have not been intensively investigated, and when they appear, their implications for measurement (or perhaps the lack of measurement) has, to my knowledge, not been explored.

Dedekind Complete Scalar Structures

DEFINITION 1.6. $\mathscr{X} = \langle X, \geqslant, R_1, R_2,... \rangle$ is said to be a *Dedekind complete scalar structure* if and only if the following three conditions are satisfied:

(1) $\mathscr{X}$ is a relational structure.

(2) $\langle X, \geqslant \rangle$ is of order type θ.

(3) $\mathscr{X}$ satisfies 1-point homogeneity and 1-point uniqueness.   ∎

Specific models of Dedekind complete scalar structures have appeared throughout the literature. The most prominent of these are models of phsyical attributes such as

length and mass. These models are structures of the form $\langle X, \succsim, \bigcirc \rangle$, where $\bigcirc$ is an associative operation on X that is strictly monotonic in each variable. Such structures and their modifications form the basis for what the literature calls "extensive measurement," a measurement process that goes back to Helmholtz (1887) and is characterized by giving numerical representations in which $\bigcirc$ is represented by the addition operation, $+$, on the positive reals. Extensive measurement gives rise to ratio scales, and these two concepts should not be confused.

Their difference becomes very apparent when one considers the work of Cohen and Narens (1979) where ratio scalability is shown to result in a generalization of extensive measurement to cases with nonassociative operations. These generalized structures are called "fundamental unit structures", and the structure $\langle Re^+, \succsim, \oplus \rangle$, where $\oplus$ is defined by $x \oplus y = x + y + x^{1/3}y^{2/3}$ for all x, y in Re^+, is an example of one of these nonassociative structures.

Dedekind complete scalar structures made their appearance in Narens (1980) under a slightly different but equivalent definition. The following important theorem is shown in Narens (1980):

THEOREM 1.2. *Suppose* $\mathscr{X} = \langle X, \succsim, R_1, R_2, ... \rangle$ *is a Dedekind complete scalar structure. Then there exists* $\mathscr{N} = \langle Re^+, \succsim, S_1, S_2, ... \rangle$ *such that the following two statements are true:*

(1) *There exists an isomorphic* $\mathscr{N}$-*representation for* $\mathscr{X}$.

(2) *The set of isomorphic* $\mathscr{N}$-*representations for* $\mathscr{X}$ *forms a ratio scale, i.e.,* (i) *for each isomorphic* $\mathscr{N}$-*representation* φ *of* $\mathscr{X}$ *and each* r *in* Re^+, $r\varphi$ *is an isomorphic* $\mathscr{N}$-*representation for* $\mathscr{X}$, *and* (ii) *for all isomorphic* $\mathscr{N}$-*representations* φ, ψ *of* $\mathscr{X}$, *there exists* s *in* Re^+ *such that* $\varphi = s\psi$.

Proof. See Theorems 2.11 and 2.6 of Narens (1980). ∎

Suppose $\mathscr{X} = \langle X, \succsim, R_1, R_2, ... \rangle$ is a relational structure and $\langle X, \succsim \rangle$ is of order type θ. Then if $\mathscr{X}$ is ratio scalable onto Re^+, then it easily follows that $\mathscr{X}$ satisfies 1-point homogeneity and 1-point uniqueness. Thus in the presence of $\mathscr{X}$ being a relational structure (a truly innocuous assumption) and $\langle X, \succsim \rangle$ being of order type θ (a natural assumption for measurement), the ratio scalability of $\mathscr{X}$ onto the positive reals is equivalent to the simultaneous assumption of 1-point homogeneity and 1-point uniqueness. It will next be shown that the simultaneous assumption of 2-point homogeneity and 2-point uniqueness for $\mathscr{X}$ is equivalent to the interval scalability of $\mathscr{X}$.

Linear Structures

DEFINITION 1.7. $\mathscr{X} = \langle X, \succsim, R_1, R_2, ... \rangle$ is said to be a *linear structure* if and only if the following three conditions are satisfied:

(1) $\mathscr{X}$ is a relational structure.

(2) $\langle X, \geqslant \rangle$ is of order type θ.

(3) $\mathscr{X}$ satisfies 2-point homogeneity and 2-point uniqueness. ∎

THEOREM 1.3. *Suppose $\mathscr{X} = \langle X, \geqslant, R_1, R_2,... \rangle$ is a linear structure. Then there exists $\mathscr{N} = \langle Re, \geqslant, S_1, S_2,... \rangle$ such that the following two statements are true:*

(1) *There exists an isomorphic $\mathscr{N}$-representation for $\mathscr{X}$.*

(2) *The set of isomorphic $\mathscr{N}$-representations for $\mathscr{X}$ forms an interval scale, i.e.,* (i) *for each isomorphic $\mathscr{N}$-representation φ of $\mathscr{X}$, $r\varphi + s$ is an isomorphic $\mathscr{N}$-representation of $\mathscr{X}$ for each r in Re^+ and s in Re, and* (ii) *for all isomorphic $\mathscr{N}$-representations φ and ψ of $\mathscr{X}$, there exist u in Re^+ and v in Re such that $\varphi = u\psi + v$.*

Proof. See Theorem 2.2. ∎

Suppose $\mathscr{X} = \langle X, \geqslant, R_1, R_2,... \rangle$ is a relational structure, $\langle X, \geqslant \rangle$ is of order type θ, and $\mathscr{X}$ is interval scalable. Then it easily follows that $\mathscr{X}$ satisfies 2-point homogeneity and 2-point uniqueness.

It should also be noted that a relational structure cannot be interval scaled onto a numerical structure with domain of discourse Re^+.

The impossibility for $n > 2$ of an n-point homogeneous, n-point unique structure

The kind of matching of homogeneity with uniqueness which appears in absolute, Dedekind complete scalar, and linear structures cannot be cannot appear in any other type of structure:

THEOREM 1.4. *There exists no structure $\mathscr{X} = \langle X, \geqslant, R_1, R_2,... \rangle$ that simultaneously satisfies the following three conditions:*

(1) $\mathscr{X}$ *is a relational structure.*

(2) $\langle X, \geqslant \rangle$ *is of order type θ.*

(3) $\mathscr{X}$ *satisfies n-point homogeneity and n-point uniqueness for some integer $n > 2$.*

Proof. See Theorem 2.3. ∎

Thus in light of Theorem 1.4, if we are to look for additional types of scales, we must look at structures that satisfy some form of infinite point homogeneity (and therefore some form of infinite point uniqueness), or look at structures where homogeneity does not match up with uniqueness. We will first look at a case of infinite point homogeneity.

Monotonic Structures

DEFINITION 1.8. Suppose $\langle X, \geqslant \rangle$ is of order type θ. Then Y is said to be *denumerably dense* in $\langle X, \geqslant \rangle$ if and only if Y is a denumerable subset of X and for

each x, z in X, if $x > z$, then for some y in Y, $x > y > z$. A subset Y of X is said to be *without endpoints* if and only if for each y in Y, there exist u, v in Y such that $u > y > v$.

Suppose $\mathscr{X} = \langle X, \geqslant, R_1, R_2, ... \rangle$ is a relational structure and $\langle X, \geqslant \rangle$ is of order type θ. Then $\mathscr{X}$ is said to satisfy *η-homogeneity* if and only if for all denumerably dense subsets without endpoints Y and Z in $\langle X, \geqslant \rangle$, there exists an automorphism α of $\mathscr{X}$ such that $\alpha(Y) = Z$. $\mathscr{X}$ is said to satisfy *η-uniqueness* if and only if for all automorphisms α, β of $\mathscr{X}$, if $\alpha(Y) = \beta(Y)$ for some denumerably dense subset without endpoints Y of $\langle X, \geqslant \rangle$, then $\alpha = \beta$. ∎

DEFINITION 1.9. $\mathscr{X} = \langle X, \geqslant, R_1, R_2, ... \rangle$ is said to be a *monotonic structure* if and only if the following three conditions are satisfied:

(1) $\mathscr{X}$ is a relational structure.

(2) $\langle X, \geqslant \rangle$ is of order type θ.

(3) $\mathscr{X}$ satisfies η-homogeneity and η-uniqueness.

THEOREM 1.5. *Suppose $\mathscr{X}$ is a monotonic structure. Then there exists $\mathscr{N} = \langle Re^+, \geqslant, S_1, S_2, ... \rangle$ such that the following two statements are true:*

(1) *There exists an isomorphic $\mathscr{N}$-representation for $\mathscr{X}$.*

(2) *The set of isomorphic $\mathscr{N}$-representations for $\mathscr{X}$ forms an ordinal scale, i.e., (i) for each isomorphic $\mathscr{N}$-representation φ of $\mathscr{X}$ and each strictly monotonic function F from Re^+ onto Re^+, $F(\varphi)$ is an isomorphic $\mathscr{N}$-representation of $\mathscr{X}$, and (ii) for all isomorphic $\mathscr{N}$-representations φ and ψ of $\mathscr{X}$, there exists a strictly monotonic function H from Re^+ onto Re^+ such that $\varphi = H(\psi)$.*

Proof. See Theorem 2.4. ∎

Suppose $\mathscr{X} = \langle X, \geqslant, R_1, R_2, ... \rangle$ is a relation structure, $\langle X, \geqslant \rangle$ is of order type θ, and $\mathscr{X}$ is ordinal scalable. Then it is not difficult to show that $\mathscr{X}$ is a monotonic structure. It is also easy to show that monotonic structures have isomorphic representations onto numerical structures that have domains of discourse Re^+ and Re.

Other Structures

DEFINITION 1.10. Let $\mathscr{X} = \langle X, \geqslant, R_1, R_2, ... \rangle$ be a relational structure, $\langle X, \geqslant \rangle$ be of order type θ, and n be a nonnegative integer. Then n is said to be the *degree of homogeneity* of $\mathscr{X}$ if and only if $\mathscr{X}$ satisfies n-point homogeneity but not $n + 1$-point homogeneity. Similarly, n is said to be the *degree of uniqueness* of $\mathscr{X}$ if and only if $\mathscr{X}$ satisfies n-point uniqueness but not $n - 1$-point uniqueness. $\mathscr{X}$ is said to have *infinite* degree of homogeneity (uniqueness) if and only if $\mathscr{X}$ does not have degree of homogeneity (uniqueness) k for some nonnegative integer k. (Monotonic structures are examples of structures with infinite degrees of homogeneity and uniqueness.) $\mathscr{X}$ is said to have a *finite* degree of homogeneity (uniqueness) if and only if $\mathscr{X}$ does not

have an infinite degree of homogeneity (uniqueness). If $\mathscr{X}$ has degree of homogeneity k and degree of uniqueness m, where k and m are nonnegative integers, then the degrees of homogeneity and uniqueness of $\mathscr{X}$ are said to *matchup* (or be *matching*) if and only if $k = m$. ∎

Let $\mathscr{X} = \langle X, \succcurlyeq, R_1, R_2, \ldots \rangle$ be a relational structure and $\langle X, \succcurlyeq \rangle$ be of order type θ. Suppose $\mathscr{X}$ has finite degrees of homogeneity and uniqueness. Since the degree of uniqueness of $\mathscr{X}$ must be at least as great as its degree of homogeneity, it follows from Theorem 1.4 that the degrees of homogeneity and uniqueness matchup if and only if they are degree 0, 1, or 2. Thus if $\mathscr{X}$ has finite degree of homogeneity and is not an absolute, Dedekind complete scalar, or linear structure, then it must have a nonmatching degree of uniqueness. Some examples of such structures with non-matching degrees will be given after the following definition.

DEFINITION 1.11. Let $\mathscr{X} = \langle X, R_0, R_1, R_2, \ldots \rangle$ be a relational structure and a be an element of X. Then a is said to be an *invariant* of $\mathscr{X}$ if and only if for each automorphism α of $\mathscr{X}$, $\alpha(a) = a$. ∎

Obviously, if a structure has an invariant, then it must have degree of homogeneity 0. Some examples of structures with degree of homogeneity 0 will now be considered. The first has degree of uniqueness 1 and no invariant elements:

EXAMPLE 1.1. Let $\mathscr{R} = \langle Re^+, \succcurlyeq, \bigcirc \rangle$, where $\bigcirc$ is defined on Re^+ as follows: for each x, y in Re^+,

$$x \bigcirc y = x + y + (xy)^{1/2}\, (2 + \sin[1/2 \log(xy)]).$$

Then by Example 3.1 of Cohen and Narens (1979), the automorphisms of $\mathscr{X}$ are functions α_n where n is an integer and for each z in Re^+,

$$\alpha_n(z) = z e^{2\pi n},$$

and from this it easily follows that $\mathscr{R}$ has no invariants, has degree of homogeneity 0, and has degree of uniqueness at least 1. From Example 3.1 and Lemma 2.2 of Cohen and Narens (1979) it follows that $\mathscr{R}$ satisfies 1-point uniqueness. Thus $\mathscr{R}$ has degree of uniqueness 1. ∎

The next example shows that some ratio scalable structures have degree of homogeneity 0 and degree of uniqueness 2:

EXAMPLE 1.2. Let $\mathscr{R} = \langle Re, \succcurlyeq, + \rangle$. It is well known that the automorphisms of $\mathscr{R}$ consist of all multiplications by positive reals, and from this it easily follows that 0 is the only invariant of $\mathscr{R}$ and that if two automorphisms of $\mathscr{R}$ agree at some nonzero point of Re, then they are identical. Thus it follows that $\mathscr{R}$ satisfies 0-point homogeneity but not 1-point homogeneity and 2-point uniqueness but not 1-point uniqueness. Now the identity function, ι, is an isomorphic $\mathscr{R}$-representation for $\mathscr{R}$,

and it easily follows that the set of all isomorphic $\mathscr{R}$-representations of $\mathscr{R}$ consists of all multiplications of ι by positive reals, i.e., that $\mathscr{R}$ is ratio scalable. ∎

The following is an example of a structure that has degree of homogeneity 0 and infinite degree of uniqueness.

EXAMPLE 1.3. For each c in Re^+, define the operation $\oplus_c$ on Re^+ as follows: for each x, y in Re^+,

$$x \oplus_c y = \frac{x + y}{1 + xy/c^2} \quad \text{if} \quad x \leqslant c \quad \text{and} \quad y \leqslant c,$$

$$x \oplus_c y = c \text{ if either } x < c \text{ and } y > c \text{ or } x > c \text{ and } y < c,$$

and

$$x \oplus_c y = \frac{x + y}{1 + xy/c^2} \quad \text{if} \quad x \geqslant c \quad \text{and} \quad y \geqslant c.$$

Then the restriction of $\oplus_c$ to the interval $(0, c)$ is the well-known "addition" operation for relativistic velocities with c being the velocity of light. It is easy to directly verify that $\oplus_c$ is an associative operation on the interval $(0, c)$ that is monotonically strictly increasing in each variable. From this is easily follows that $\mathscr{L}_c = \langle (0, c), \geqslant, \oplus_c \rangle$ is what Chapter 3 of Krantz, Luce, Suppes, and Tversky (1971) calls an "extensive structure," and it also easily follows from Theorem 3 of that chapter that $\mathscr{L}_c$ is a Dedekind complete scalar structure. As is the usual mathematical practice, let $(c, \infty) = \{x \mid x \in Re^+ \text{ and } x > c\}$. Again it is easy to show by direct verification that $\oplus_c$ is an associative operation on (c, ∞) that is monotonically strictly decreasing in each variable. From this it follows by Chapter 3 of Krantz *et al.* (1971) that $\langle (c, \infty), \leqslant, \oplus_c \rangle$ is an extensive structure, and from this it immediately follows that $\mathscr{R}_c = \langle (c, \infty), \geqslant, \oplus_c \rangle$ is a Dedekind complete scalar structure. Let α be an automorphism of $\mathscr{L}_c$, β be the identity function on $\{c\}$, and γ be an automorphism of $\mathscr{R}_c$. Then it is easy to show that $\alpha \cup \beta \cup \gamma$ is an automorphism of $\mathscr{S}_c = \langle Re^+, \geqslant, \oplus_c \rangle$ and that all automorphisms of $\mathscr{S}_c$ are of this form. $\mathscr{S}_c$ has 0 degree of homogeneity since it has c as an invariant element. Let α_1 and α_2 be automorphisms of $\mathscr{L}_c$ such that $\alpha_1 \neq \alpha_2$, γ be the identity function on $\{c\}$, and β be an automorphism of $\mathscr{R}_c$. Then $\alpha_1 \cup \gamma \cup \beta$ and $\alpha_2 \cup \gamma \cup \beta$ are distinct automorphisms of $\mathscr{S}_c$ that agree on all elements of (c, ∞), and from this it follows that $\mathscr{S}_c$ has infinite degree of uniqueness. Note that if two automorphisms of $\mathscr{S}_c$ agree at an element $<c$ and an element $>c$, then they are identical. ∎

Structures with invariant elements often present technical difficulties for measurement theory. For example, consider the measurement of a structure $\mathscr{S}$ that for some c in Re^+ is isomorphic to $\mathscr{S}_c$ as defined in Example 1.3. Each $\mathscr{S}_c$ (c in Re^+) is a reasonable candidate for a numerical representing structure for $\mathscr{S}$ since for each d, e in Re^+, $\mathscr{S}_d$ and $\mathscr{S}_e$ are isomorphic by the function $f(x) = (e/d)x$. Thus a

particular one, say, $\mathscr{S}_{10}$, can be picked as the numerical representing structure. However, to my point of view this produces an unwanted consequence: namely, the invariant element of $\mathscr{S}$ under each permissible assignment is assigned the number 10, and clearly there is no special relationship between the number 10 and the invariant element of $\mathscr{S}$; rather this relationship results from the arbitrary selection of $\mathscr{S}_{10}$ as the numerical representing structure. The way I suggest to proceed with situations like this is to generalize the measurement process so that the representations of a structure are onto members of a set of structures rather than onto a single structure. In the present case, a proper measurement of $\mathscr{S}$ would be any isomorphism onto $\mathscr{S}_c$ for some c in Re^+. In this situation, any two representations φ, ψ of $\mathscr{S}$ would then be represented by the formula $\varphi(x) = \alpha[\beta(\psi(x))]$, where for some $\mathscr{S}_c$ and $\mathscr{S}_d$, β is an automorphism of $\mathscr{S}_d$ and α is the isomorphism of $\mathscr{S}_d$ onto $\mathscr{S}_c$ given by the formula $\alpha(y) = (c/d) y$.

The following definition gives a general formulation of representation into a set of structures:

DEFINITION 1.12. Suppose $\mathscr{X}$ is a relational structure and $\mathbf{Y}$ is a nonempty set of relational structures such that each element of $\mathbf{Y}$ has domain of discourse a subset of the reals. Then φ is said to be $\mathbf{Y}$-*representation* for $\mathscr{X}$ if and only if φ is a homomorphism of $\mathscr{X}$ onto some element of $\mathbf{Y}$. ∎

Suppose $\mathscr{X}$ is a totally ordered relational structure, $\mathbf{Y}$ is a set of relational structures, and a $\mathbf{Y}$-representation for $\mathscr{X}$ exists. Since $\mathscr{X}$ is totally ordered, all homomorphisms of $\mathscr{X}$ are one-to-one functions and thus isomorphisms, and from this it easily follows that all elements of $\mathbf{Y}$ are isomorphic. Let φ be a $\mathbf{Y}$-representation for $\mathscr{X}$. Then it is easy to show that for each $\mathbf{Y}$-representation ψ of $\mathscr{X}$ there exist an isomorphism α of $\varphi(\mathscr{X})$ onto $\psi(\mathscr{X})$ so that for each x in the domain of discourse of $\mathscr{X}$, $\psi(x) = \alpha[\varphi(x)]$.

Suppose $\mathscr{X}$ is a relational structure and $\mathscr{N}$-representations for $\mathscr{X}$ exist. Let $\mathbf{Y} = \{\varphi(\mathscr{X}) \mid \varphi$ is an $\mathscr{N}$-representation for $\mathscr{X}\}$. Then the $\mathscr{N}$-representations and $\mathbf{Y}$-representations for $\mathscr{X}$ coincide. Thus the concept of $\mathscr{N}$-representation (Definition 1.2) is a special case of the concept of $\mathbf{Y}$-representation (Definition 1.12).

A serious problem for measurement theory—and one that in my opinion has not been satisfactorily dealt with—is the selection of "correct" set of scales for measurement in an empirical context from the set of possible candidate scales. The traditional proposed solution to this problem has been to choose a numerical representing structure, $\mathscr{N}$, for the empirical structure under consideration, $\mathscr{X}$, and consider the $\mathscr{N}$-representations for $\mathscr{X}$ as the "correct" scales. But traditionally, very little attention has been given to *justifying* the choice of $\mathscr{N}$ over other possible numerical representing structures, e.g., numerical structures isomorphic to $\mathscr{N}$. Furthermore, as we have seen above, representing into a single numerical structure is sometimes too constraining. However in some situations even $\mathbf{Y}$-representations (Definition 1.12) may be too constraining since if φ is a $\mathbf{Y}$-representation for $\mathscr{X}$ and α is an automorphism of $\varphi(\mathscr{X})$, then ψ defined by $\psi(x) = \alpha[\varphi(x)]$ is also a $\mathbf{Y}$-

representation, and it is conceivable that in some situations this may give rise to too many scales for the measurement process to be effectively carried out, particularly if the measurement of $\mathscr{X}$ is the measurement of a variable that interacts with other variables of the empirical context under consideration.

The idea behind the construction in Example 1.3 is very general and can be extended to other situations. The gist of it is to select a point in Re^+ and consider the two open intervals determined by it. On each interval a totally ordered relational structure is constructed. These relational structures may or may not be naturally related. (In Example 1.3 they are strongly related.) However, they must be of the same similarity type. (Two relational structures $\langle X, S_1, S_2, \ldots \rangle$ and $\langle Y, T_1, T_2, \ldots \rangle$ are said to be of the *same similarity type* if and only if for all n in I^+, S_i is an n-ary relation if and only if T_i is an n-ary relation.) Another relational structure of the same similarity type is then constructed on Re^+ that is characterized by having only trivial interactions occurring between elements of opposite intervals or between the selected element and other elements of Re^+. This kind of lack of interaction basically reduces the relational structure on Re^+ to a disjoint union of the two relational structures defined on the intervals, and thus autmorphisms of the larger structure are basically decomposable into automorphisms of the smaller ones and vice versa. This allows for the construction of structures with automorphisms behaving on pieces of the structure in predetermined ways. Of course this kind of construction generalizes to cases with any number of intervals, and a great variety of relational structures with specific automorphism properties can be constructed in this manner by the appropriate use of absolute, Dedekind complete scalar, interval, and monotonic structures as component parts.

Structures with infinite degrees of homogeneity and uniqueness that are not monotonic structures can be constructed, but because of the apparent adequacy of monotonic structures for the infinite homogeneous case and the current lack of applications in science of other infinitely homogeneous structures, the description and development of these kinds of structures will be omitted.

Discussion

Homogeneity and uniqueness have been shown to provide a useful basis for classifying and specifying measurement structures. Indeed, they provide a means for formulating necessary and sufficient conditions in the general case of relational structures for the traditional types of ordered scalings—ratio, interval, and ordinal. Furthermore, we have shown that a characteristic property of structures with these traditional scalings—having matching degrees of homogeneity and uniqueness— extends to other cases with finite degrees of homogeneity to only those degenerate cases of a scale type consisting of a single scale. (This may be part of the reason why scientists have not employed other types of ordered scales.) Although I consider the m-point homogeneity, n-point uniqueness classification of measurement structures an improvement, it has obvious shortcomings, particularly in dealing with structures with invariant elements, and more refined concepts of homogeneity and uniqueness are needed to achieve a better and more systematic classification.

The approach taken in this paper is to describe measurement structures in terms of their automorphisms. The rationale for this is that the most important measurement-theoretic properties of a scale type are reflected in the automorphisms of any empirical structure scalable by that type. However, the concept of "scale type" given requires that the *whole* structure be given a numerical representation, and for some applications this may be too restrictive. (For example, in geometry local coordinate systems are employed that only partially represent the space.) Of course, the abandonment of the requirement of representing the whole structure will limit the usefulness of automorphisms as a means of classifying measurement structures. Section 3 of Narens (1980) discusses this problem and indicates some promising generalizations of the concept "automorphism."

In summary, I think the description presented here of measurement structures in terms of their automorphisms has been fruitful and has provided insights into the possible range of measurement structures. However, there are still problems, and from my view, the most important of these revolve around the concept of "representation": at this time we just do not have a very clear idea as to what "representations" should be or how to select the "correct" one or set of ones.

PART 2: THEOREMS

DEFINITION 2.1. $\langle X, \geqslant, A \rangle$ is said to be a *structure of linear automorphisms* if and only if X is nonempty, $\geqslant$ is a binary relation on X, $\langle A, * \rangle$ is a subgroup of automorphisms of $\langle X, \geqslant \rangle$, and the following three conditions hold:

   (1) $\langle X, \geqslant \rangle$ is of order type θ.

   (2) A satisfies 2-point homogeneity.

   (3) A satisfies 2-point uniqueness. ∎

LEMMA 2.1.   *Let* $\langle X, \geqslant, A \rangle$ *be a structure of linear automorphisms. Suppose* α, β *are in* A, a, b *are in* X, $a < b$, $\alpha(a) < \beta(a)$, *and* $\beta(b) < \alpha(b)$. *Then for some* u *in* X, $a < u < b$ *and* $\alpha(u) = \beta(u)$.

Proof.   By Theorem 1.1, let φ be an isomorphic representation of $\langle X, \geqslant \rangle$ onto $\langle Re^+, \geqslant \rangle$. Let a', b', α', β' be the images under φ of a, b, α, β respectively. Since α' and β' are automorphisms of $\langle Re^+, \geqslant \rangle$, they are order preserving functions from Re^+ onto Re^+ and therefore are continuous. Consider the function $\gamma = \beta' - \alpha'$. γ is continuous, $\gamma(a') > 0$, and $\gamma(b') < 0$. Thus by the intermediate value theorem of analysis, there exists u' in Re^+ such that $a' < u' < b'$ and $\gamma(u') = 0$, i.e., $\alpha'(u') = \beta'(u')$. Letting $u = \varphi^{-1}(u')$, it follows by isomorphism that $a < u < b$ and $\alpha(u) = \beta(u)$. ∎

DEFINITION 2.2.   Let $\langle X, \geqslant, A \rangle$ be a structure of linear automorphisms. Define $\succeq$ on A as follows: for each α, β in A, $\alpha \succeq \beta$ if and only if there exists y in X such that for all $x \geqslant y$, $\alpha(x) \geqslant \beta(x)$. ∎

LEMMA 2.2. *Let $\langle X, \geqslant, A \rangle$ be a structure of linear automorphisms. Then $\geqslant$ is a total ordering on A.*

Proof. Let α, β, and γ be arbitrarily elements of A.

Transitivity. If $\alpha \geqslant \beta$ and $\beta \geqslant \gamma$, then it immediately follows from Definition 2.2 that $\alpha \geqslant \gamma$.

Connectivity. To show a contradiction, suppose that neither $\alpha \geqslant \beta$ nor $\beta \geqslant \alpha$. Then for each y in X, there are x, z in X such that $y \leqslant x < z$, $\alpha(x) < \beta(x)$, and $\beta(z) < \alpha(z)$, which by Lemma 2.1 implies that for each y in X there exists $u > y$ such that $\alpha(u) = \beta(u)$, i.e., α and β agree on infinitely many elements of X, which by 2-point uniqueness yields $\alpha = \beta$, and this contradicts the hypothesis that it is not the case that $\alpha \geqslant \beta$.

Antisymmetry. Suppose $\alpha \geqslant \beta$ and $\beta \geqslant \alpha$. Then it follows from Definition 2.2 that there is an x in X such that for all a, b in X if $x < a < b$, then $\alpha(a) \leqslant \beta(a)$ and $\beta(b) \leqslant \alpha(b)$, and this (by using Lemma 2.1 if necessary) implies that $\alpha(u) = \beta(u)$ for infinitely many u in X, which in turn by 2-point uniqueness implies $\alpha = \beta$. ∎

LEMMA 2.3. *Let $\langle X, \geqslant, A \rangle$ be a structure of linear automorphisms. Suppose a, b are elements of X, α, β are elements of A, $a < b$, $\alpha(a) < \beta(a)$, and $\beta(b) < \alpha(b)$. Then $\alpha > \beta$.*

Proof. By Lemma 2.1, let u in X be such that $a < u < b$ and $\alpha(u) = \beta(u)$. To show a contradiction, suppose it is not the case that $\alpha > \beta$. Then by Lemma 2.2, $\beta \geqslant \alpha$, and thus by Definition 2.2 we can find c in X such that $b < c$ and $\alpha(c) \leqslant \beta(c)$. If $\alpha(c) = \beta(c)$, let $v = c$, and if $\alpha(c) < \beta(c)$, by Lemma 2.1, let v in X be such that $b < v < c$ and $\alpha(v) = \beta(v)$. Then $\alpha(v) = \beta(v)$, and thus by 2-point uniqueness, $\alpha = \beta$, which is impossible since $\alpha(a) \neq \beta(a)$. ∎

LEMMA 2.4. *Let $\langle X, \geqslant, A \rangle$ be a structure of linear automorphisms. Then $\langle A, \geqslant, * \rangle$ is a totally ordered group.*

Proof. It is well-known that $\langle A, * \rangle$ is a group, and by Lemma 2.2, $\geqslant$ is a total ordering on A. Suppose α, β, and γ are arbitrary elements of A.

We will first show $\alpha \geqslant \beta$ iff $\alpha * \gamma \geqslant \beta * \gamma$. Suppose $\alpha \geqslant \beta$. By Definition 2.2, let y_0 in X be such that for all $x \geqslant y_0$, $\alpha(x) \geqslant \beta(x)$. Since γ is order preserving and is onto X, let y_1 in X be such that for all $x \geqslant y_1$, $\gamma(x) \geqslant y_0$. Then for all $x \geqslant y_1$, $\alpha * \gamma(x) \geqslant \beta * \gamma(x)$, and thus by Definition 2.2, $\alpha * \gamma \geqslant \beta * \gamma$. Now suppose $\alpha * \gamma \geqslant \beta * \gamma$. By Definition 2.2, let y_2 in X be such that for all $x \geqslant y_2$, $\alpha * \gamma(x) \geqslant \beta * \gamma(x)$. Let $y_3 = \gamma(y_2)$. Then since γ is order preserving and is onto X, for each $x \geqslant y_3$ there exists $u \geqslant y_2$ such that $x = \gamma(u)$ and thus

$$\alpha(x) = \alpha * \gamma(u) \geqslant \beta * \gamma(u) = \beta(x),$$

which by Definition 2.2 yields $\alpha \geqslant \beta$.

We will now show $\alpha \geqslant \beta$ iff $\gamma * \alpha \geqslant \gamma * \beta$. Suppose $\alpha \geqslant \beta$. By Definition 2.2, let y_0 in X be such that for each $x \geqslant y_0$, $\alpha(x) \geqslant \beta(x)$. Then, since γ is order preserving, for each $x \geqslant y_0$, $\gamma * \alpha(x) \geqslant \gamma * \beta(x)$, and thus by Definition 2.2, $\gamma * \alpha \geqslant \gamma * \beta$. Now suppose $\gamma * \alpha \geqslant \gamma * \beta$. Let y_1 in X be such that for all $x \geqslant y_1$, $\gamma * \alpha(x) \geqslant \gamma * \beta(x)$. Then, since γ is order preserving, $\alpha(x) \geqslant \beta(x)$ for all $x \geqslant y_1$, and thus by Definition 2.2, $\alpha \geqslant \beta$. ∎

DEFINITION 2.3.   Let $\langle X, \geqslant, A \rangle$ be a structure of linear automorphisms, $\alpha \in A$, and $a \in X$. Then α is said to be a *dilation at* a if and only if $\alpha(a) = a$. α is said to be a *dilation* if and only if α is a dilation at b for some b in X. ∎

LEMMA 2.5.   *Let $\langle X, \geqslant, A \rangle$ be a structure of linear automorphisms, α and β be elements of A, and a be an element of X. Suppose α and β are dilations at a and $\alpha > \beta$. Then $\alpha(x) > \beta(x)$ for all $x > a$.*

Proof. To show a contradiction, suppose b in X is such that $a < b$ and $\alpha(b) \leqslant \beta(b)$. If $\alpha(b) = \beta(b)$, then by 2-point uniqueness $\alpha = \beta$, contradicting the hypothesis $\alpha > \beta$. Thus $\alpha(b) < \beta(b)$. Since $\alpha > \beta$, by Definition 2.2, let c in X be such that $b < c$ and $\beta(c) < \alpha(c)$. Then by Lemma 2.1, let u in X be such that $b < u < c$ and $\alpha(u) = \beta(u)$. Then by 2-point uniqueness, $\alpha = \beta$, contradicting the hypothesis $\alpha > \beta$. ∎

Recall that ι is the identity element of A.

LEMMA 2.6.   *Let $\langle X, \geqslant, A \rangle$ be a structure of linear automorphisms. Suppose a and b are in X, $a < b$, α and β are in A, $\iota < \alpha < \beta$, α is a dilation at a and β is a dilation at b. Then $\beta(a) < a$.*

Proof. $\beta(a) \neq a$, since if $\beta(a) = a$, then by 2-point uniqueness $\beta = \iota$, contrary to hypothesis. It is also not the case that $\beta(a) > a$, for if $\beta(a) > a = \alpha(a)$, then by Lemma 2.5,

$$\alpha(b) > \iota(b) = b = \beta(b),$$

which by Lemma 2.1 implies that $\alpha(u) = \beta(u)$ for some u in X for which $a < u < b$, but since $\alpha < \beta$, for some sufficiently large c in X, $\beta(c) > \alpha(c)$, and this together with $\alpha(b) > \beta(b)$ and Lemma 2.1 yields $\alpha(v) = \beta(v)$ for some v such that $b < v < c$, which in turn by 2-point uniqueness yields $\alpha = \beta$, which contradicts the hypothesis $\alpha < \beta$. Since $\geqslant$ is a total ordering on X and $\beta(a) \neq a$ and it is not the case that $\beta(a) > a$, it follows that $\beta(a) < a$. ∎

DEFINITION 2.4.   Let $\langle X, \geqslant, A \rangle$ be a structure of linear automorphisms. Then, by definition, for each a in X, let

$$A_a = \{\alpha \mid \alpha \text{ is a dilation at } a\}$$

and $\mathscr{S}_a = \langle A_a, \geqslant, * \rangle$.

LEMMA 2.7. *Let $\langle X, \geqslant, A \rangle$ be a structure of linear automorphisms and $\mathscr{G} = \langle A, \geqslant, * \rangle$. Then $\mathscr{G}$ is a nonarchimedean totally ordered group.*

Proof. By Lemma 2.4, $\mathscr{G}$ is a totally ordered group. It is well-known that all Archimedean totally ordered groups are commutative. Thus to show $\mathscr{G}$ is nonarchimedean it is sufficient to show it is noncommutative. Let a, b, c, d be elements of X such that $a < b < c < d$. By 2-point homogeneity, let α, β be elements of A such that $\alpha(a) = a$, $\alpha(b) = c$, $\beta(b) = b$, $\beta(c) = d$. Then

$$\alpha * \beta(b) = c < d = \beta * \alpha(b),$$

and thus $\alpha * \beta \neq \beta * \alpha.$ ∎

LEMMA 2.8. *Let $\langle X, \geqslant, A \rangle$ be a structure of linear automorphisms. Then for each a in X, $\mathscr{G}_a$ is a Dedekind complete, dense, totally ordered group.*

Proof. It is immediate that $\mathscr{G}_a$ is a nontrivial, totally ordered group.

To show $\mathscr{G}_a$ is dense, it is sufficient to show for all η and σ in A_a such that $\iota < \eta < \sigma$, there exists ν in A_a such that $\eta < \nu < \sigma$. Thus let η and σ be elements of A_a such that $\iota < \eta < \sigma$, and let d be an element of X such that $a < d$ and $a < \eta(d) < \sigma(d)$. Since $\langle X, \geqslant \rangle$ is of order type θ, let e in X be such that $\eta(d) < e < \sigma(d)$. Then by 2-point homogeneity let ν in A be such that $\nu(a) = a$ and $\nu(d) = e$. Then ν is in A_a, and by Lemma 2.5, $\eta < \nu < \sigma$, i.e., $\mathscr{G}_a$ is dense.

To show Dedekind completeness, let S be a nonempty subset of A_a and β be an element of A_a such that $\beta \geqslant \alpha$ for each α in S. Let b be an element of X such that $b > a$. Then by Lemma 2.5, $\beta(b) \geqslant \alpha(b)$ for each α in S. By the Dedekind completeness of $\langle X, \geqslant \rangle$, let

$$y = \text{l.u.b.} \{ \alpha(b) \mid \alpha \in S \}.$$

By 2-point homogeneity, let γ in A_a be such that $\gamma(b) = y$. Let δ be an arbitrary element A_a that is an upper bound of S, i.e., δ is in A_a and $\delta \geqslant \alpha$ for each α in S. Then for all α in S,

$$\delta(b) \geqslant y = \gamma(b) \geqslant \alpha(b). \tag{2.1}$$

Thus if we can show that $\delta \geqslant \gamma$, then we have shown that γ is the least upper bound of S and thus that $\mathscr{G}_a$ is Dedekind complete. To show that $\delta \geqslant \gamma$, assume the contrary, i.e., $\gamma > \delta$. By Definition 2.2, let c be an element of X such that $c > b$ and $\gamma(c) > \delta(c)$. Now $\delta(b) \neq \gamma(b)$, since if $\delta(b) = \gamma(b)$ then by 2-point uniqueness $\delta = \gamma$, contradicting $\gamma > \delta$. Thus by Eq. (2.1), $\delta(b) > \gamma(b)$. Therefore by Lemma 2.1, let u in X be such that $b < u < c$ and $\delta(u) = \gamma(u)$. Then by 2-point uniqueness $\delta = \gamma$, a contradiction. ∎

LEMMA 2.9. *Let $\langle X, \geqslant, A \rangle$ be a structure of linear automorphisms. Then for each a in X, $\mathscr{G}_a$ is Archimedean.*

Proof. It is well-known that all Dedekind complete, totally ordered groups are Archimedean. Thus by Lemma 2.8, $\mathscr{S}_a$ is Archimedean. ∎

LEMMA 2.10. *Let* $\langle X, \geqslant, A \rangle$ *be a structure of linear automorphisms. Suppose* α *and* β *are dilations of* A *such that* $\alpha > \iota$ *and* $\beta > \iota$. *Then for some* m, n *in* I^+,

$$\alpha^m > \beta \quad and \quad \beta^n > \alpha.$$

Proof. Since $\geqslant$ is a total ordering on A, either $\alpha \geqslant \beta$ or $\beta \geqslant \alpha$. Without loss of generality, suppose $\alpha \geqslant \beta$. Then $\alpha^2 > \beta$. Thus we need to only show $\beta^n > \alpha$ for some n in I^+. If $\beta = \alpha$, this immediately follows since then $\beta^2 > \alpha$. Thus suppose $\alpha > \beta$.

Let a, b be elements X such that α is a dilation at a and β is a dilation at b. If $a = b$, then since $\mathscr{S}_a = \mathscr{S}_b$ is Archimedean, $\beta^n > \alpha$ for some n in I^+. Thus assume $a \neq b$. There are two cases to consider.

Case 1. $a < b$. Then since $\alpha > \iota$, by Lemma 2.5, $\alpha(b) > \iota(b) = b$. Thus $\beta(b) < \alpha(b)$. Let c, d be elements of X such that $b < c$ and $\alpha(c) < d$. By 2-point homogeneity, lt γ in A be such that $\gamma(b) = b$ and $\gamma(c) = d$. Then γ is in A_b and

$$\gamma(b) < \alpha(b) \quad and \quad \alpha(c) < \gamma(c),$$

which by Lemma 2.3 yields $\gamma > \alpha$. Since by Lemma 2.9, $\mathscr{S}_b$ is Archimedean, let n in I^+ be such that $\beta^n > \gamma$. Then $\beta^n > \alpha$.

Case 2. $b < a$. By Lemma 2.6, $\alpha(b) < b$. Since α is onto X, let z in X be such that $\alpha(z) = b$. Since $\alpha(b) < b = \alpha(z)$ and α is order preserving, $b < z$. Let y in X be such that $y < b$. By 2-point homogeneity, let γ in A be such that $\gamma(y) = y$ and $\gamma(b) = z$. $\gamma \neq \iota$ since $\gamma(b) = z > b$. Also is is not the case that $\gamma < \iota$, since then $\gamma(c) < \iota(c) = c$ for some c in X such that $b < c$ and since $\iota(b) = b < z = \gamma(b)$, it would then follow from Lemma 2.1 that $\gamma(u) = \iota(u) = u$ for some $b < u < c$, which by 2-point uniqueness implies that $\gamma = \iota$, which contradicts $\gamma < \iota$. Thus, since $\geqslant$ is a total ordering on X, $\gamma > \iota$. Thus $\alpha * \gamma > \iota$, and since $\alpha * \gamma(b) = \alpha(z) = b$, $\alpha * \gamma$ is in A_b. Since by Lemma 2.9, $\mathscr{S}_b$ is Archimedean, let n in I^+ be such that $\beta^n > \alpha * \gamma$. Then $\beta^n > \alpha$ since $\alpha * \gamma > \alpha$. ∎

LEMMA 2.11. *Let* $\langle X, \geqslant, A \rangle$ *be a structure of linear automorphisms. Then for each* α *in* A, *if* $\alpha > \iota$ *then there exists a dilation* β *in* A *such that* $\beta > \alpha$.

Proof. Let α be an arbitrary element of A such that $\alpha > \iota$. Since $\alpha > \iota$, let a in X be such that $a < \alpha(a)$. Let b in X be such that $a < b$ and let c be an element of X such that $\alpha(b) < c$. By 2-point homogeneity, let β in A be such that $\beta(a) = a$ and $\beta(b) = c$. Then β is a dilation in A, and $\beta(a) < \alpha(a)$ and $\alpha(b) < \beta(b)$. By Lemma 2.3, $\beta > \alpha$. ∎

DEFINITION 2.5. Let $\langle X, \geqslant, A \rangle$ be a structure of linear automorphisms. α in A is said to be a *positive infinitesimal* if and only if $\alpha > \iota$ and for each dilation $\beta > \iota$ and each n in I^+, $\beta > \alpha^n$. α in A is said to be *infinitesimal* if and only if α is a positive

infinitesimal or $\alpha = \iota$ or α^{-1} is a positive infinitesimal. γ in A is said to be *non-infinitesimal* if and only if γ is not infinitesimal. ∎

LEMMA 2.12. *Let* $\langle X, \geqslant, A \rangle$ *be a structure of linear automorphisms and α be an element of A. Then α is a positive infinitesimal if and only if $\alpha > \iota$ and for some dilation β in A, $\beta > \alpha^n$ for each n in I^+.*

Proof. Suppose α is a positive infinitesimal. Then it is immediate from Definition 2.5 that for some dilation β of A, $\beta > \alpha^n$ for all n in I^+.

Suppose α is in A, $\alpha > \iota$, β is a dilation in A, and $\beta > \alpha^n$ for all n in I^+. Then $\beta > \iota$. Let γ be an arbitrary dilation in A such that $\gamma > \iota$. By Lemma 2.10, let k in I^+ be such that $\gamma^k > \beta$. Then for each n in I^+, $\gamma^k > \beta > \alpha^{k \cdot n}$, from which it follows that $\gamma > \alpha^n$ for all n in I^+, which by Definition 2.5 implies that α is a positive infinitesimal. ∎

LEMMA 2.13. *Let* $\langle X, \geqslant, A \rangle$ *be a structure of linear automorphisms and α, β be infinitesimal elements of A. Then α^{-1} and $\alpha * \beta$ are infinitesimal.*

Proof. It is immediate from Definition 2.5 that α^{-1} is an infinitesimal. There are three cases to consider.

Case 1. $\alpha * \beta = \iota$. Then by Definition 2.5, $\alpha * \beta$ is infinitesimal.

Case 2. $\alpha * \beta > \iota$. Then either $\alpha \geqslant \beta$ or $\beta \geqslant \alpha$. Without loss of generality, assume $\alpha \geqslant \beta$. Then $\alpha^2 \geqslant \alpha * \beta$, and since α is a positive infinitesimal,

$$\gamma > \alpha^{2n} = (\alpha^2)^n$$

for each dilation γ in A such that $\gamma > \iota$ and each n in I^+, and thus $\gamma > (\alpha * \beta)^n$ for each dilation $\gamma > \iota$ and each n in I^+. Thus $\alpha * \beta$ is infinitesimal.

Case 3. $\alpha * \beta < \iota$. Then $(\alpha * \beta)^{-1} = \beta^{-1} * \alpha^{-1} > \iota$, and thus by Case 2, $(\alpha * \beta)^{-1}$ is infinitesimal, which implies $\alpha * \beta$ is infinitesimal. ∎

LEMMA 2.14. *Let* $\langle X, \geqslant, A \rangle$ *be a structure of linear automorphisms. Then there exists a positive infinitesimal in A.*

Proof. By Lemma 2.7, $\mathscr{S} = \langle A, \geqslant, * \rangle$ is nonarchimedean. Thus let α, β be positive elements of $\mathscr{S}$ such that $\alpha^n < \beta$ for all n in I^+. By Lemma 2.11, let γ be a dilation in A such that $\beta < \gamma$. Then $\alpha^n < \gamma$ for all n in I^+, which by Lemma 2.12 yields that α is a positive infinitesimal. ∎

LEMMA 2.15. *Let* $\langle X, \geqslant, A \rangle$ *be a structure of linear automorphisms and α, β be elements of A. Suppose α is infinitesimal and β is noninfinitesimal. Then $\beta^{-1} * \alpha * \beta$ is infinitesimal.*

Proof. If $\beta^{-1}\alpha\beta = \iota$, then $\beta^{-1}\alpha\beta$ is infinitesimal. So suppose $\beta^{-1}\alpha\beta \neq \iota$. By

Lemma 2.11, let γ be a dilation in A such that $\gamma > \beta$, β^{-1}, α. Then, since α is infinitesimal, $\gamma > \alpha^{3n}$ for each n in I^{+}. Thus for each n in I^{+},

$$\gamma^3 > \beta^{-1} \alpha^{3n} \beta = (\beta^{-1} \alpha \beta)^{3n},$$

and therefore

$$\gamma > (\beta^{-1} \alpha \beta)^n.$$

Similarly, for each n in I^{+}

$$\gamma > (\beta^{-1} \alpha^{-1} \beta)^n = [(\beta^{-1} \alpha \beta)^{-1}]^n.$$

Since either $\beta^{-1} \alpha \beta > \iota$ or $(\beta^{-1} \alpha \beta)^{-1} > \iota$, we have shown (using Lemmas 2.12 and 2.13) that both $\beta^{-1} \alpha \beta$ and $(\beta^{-1} \alpha \beta)^{-1}$ are infinitesimal. ∎

LEMMA 2.16. *Let* $\langle X, \geqslant, A \rangle$ *be a structure of linear automorphisms and* α, β *be elements of* A. *Suppose* α, β *are infinitesimals and* $\alpha(x) = \beta(x)$ *for some* x *in* X. *Then* $\alpha = \beta$.

Proof. Let x in X be such that $\alpha(x) = \beta(x)$. Then $\beta^{-1} * \alpha(x) = x$. Thus $\beta^{-1} * \alpha$ is a dilation. By Lemma 2.13, $\beta^{-1} * \alpha$ is infinitesimal. Since ι is the only infinitesimal dialtion, $\beta^{-1} * \alpha = \iota$, and thus $\alpha = \beta$. ∎

LEMMA 2.17. *Let* $\langle X, \geqslant, A \rangle$ *be a structure of linear automorphisms, a be an element of* X, *and* α *in* A *be a positive infinitesimal. Then for each* x *in* X, *if* $x > a$, *then there exists* β *in* A_a *such that* $\beta^{-1} * \alpha * \beta(a) = x$.

Proof. Let x be an arbitrary element of X such that $x > a$. Since $\alpha > \iota$, by Lemma 2.16, $\alpha(a) > a$. If $\alpha(a) = a$, then $\alpha = \iota$ by Lemma 2.16. If $\alpha(a) < a$, then since $\alpha > \iota$, $\alpha(b) > b$ for some $b > a$, which by Lemma 2.1 yields $\alpha(u) = u$ for some u in X, which by Lemma 2.16 yields $\alpha = \iota$, which is impossible. Thus $\alpha(a) > a$. Therefore by 2-point uniqueness, let β in A be such that $\beta(x) = \alpha(a)$ and $\beta(a) = a$. Then β is an A_a, and $\beta^{-1} * \alpha * \beta(a) = x$. ∎

LEMMA 2.18. *Let* $\langle X, >, A \rangle$ *be a structure of linear automorphisms. Then for each* x, y *in* X, *there exists an infinitesimal* α *in* A *such that* $\alpha(x) = y$.

Proof. Let x, y be arbitrary elements of X. There are three cases to consider:

Case 1. $x < y$. By Lemma 2.14, let γ be a positive infinitesimal. By Lemma 2.17, let β be an element of A_x such that $\beta^{-1} * \gamma * \beta(x) = y$. Then by Lemma 2.15, $\beta^{-1} * \gamma * \beta$ is an infinitesimal.

Case 2. $x = y$. Then $\iota(x) = y$ and ι is an infinitesimal.

Case 3. $x > y$. Applying Case 1 to $y < x$, we can find an infinitesimal α such that $\alpha(y) = x$. Then $\alpha^{-1}(x) = y$ and by Lemma 2.13, α^{-1} is an infinitesimal. ∎

LEMMA 2.19. *Let* $\langle X, \geqslant, A \rangle$ *be a structure of linear automorphisms. Suppose* α, β *are infinitesimal elements of A. Then* $\alpha(x) \geqslant \beta(x)$ *for each x in X if and only if* $\alpha \geqslant \beta$.

Proof. Suppose $\alpha(x) \geqslant \beta(x)$ for each x in X. Then by Definition 2.2, $\alpha \geqslant \beta$.

Suppose $\alpha \geqslant \beta$. We will show $\alpha(x) \geqslant \beta(x)$ for each x in X by contradiction. Suppose a in X is such that $\alpha(a) < \beta(a)$. Then $\alpha \neq \beta$ and thus $\alpha > \beta$. Therefore by Definition 2.2 let b in X be such that $a < b$ and $\beta(b) < \alpha(b)$. Then by Lemma 2.1, let u in X be such that $\alpha(u) = \beta(u)$. Then by Lemma 2.16, $\alpha = \beta$, which is impossible since $\alpha(x) < \beta(a)$.

LEMMA 2.20. *Let* $\langle X, \geqslant, A \rangle$ *be a structure of linear automorphisms and M be the set of infinitesimal elements of A. Then the following three statements are true*:

(1) $\langle M, \geqslant \rangle$ *is of order type* θ.

(2) $\langle M, \geqslant, * \rangle$ *is a Dedekind complete totally ordered group.*

(3) $\langle M, * \rangle$ *is commutative.*

Proof. (1) Let a be an element of X and let f be the function from A into X defined by $f(\alpha) = \alpha(a)$ for all α in A. Then by Lemma 2.18 and 2.19, f is an isomorphism of $\langle A, \geqslant \rangle$ onto $\langle X, \geqslant \rangle$, and since $\langle X, \geqslant \rangle$ is of order type θ, it then follows that $\langle A, \geqslant \rangle$ is of order type θ.

(2) $\langle M, * \rangle$ is a group by Lemma 2.13, and $\langle M, \geqslant, * \rangle$ is therefore a totally ordered subgroup of $\langle A, \geqslant, * \rangle$ by Lemma 2.4. $\langle M, \geqslant \rangle$ is Dedekind complete by Statement 1.

(3) By Statement 2, $\langle M, \geqslant, * \rangle$ is a Dedekind complete totally ordered group, and it is well-known that all such groups are commutative. ∎

Some of the more important facts of the previous lemmas are summarized in the following theorem:

THEOREM 2.1. *Suppose* $\langle X, \geqslant, A \rangle$ *is a structure of linear automorphisms and M is the set of infinitesimals of A. Then the following three statements are true*:

(1) $\langle M, * \rangle$ *is commutative.*

(2) *M satisfies 1-point homogeneity and 1-point uniqueness.*

(3) *For each* α *in M and each* β *in* $A - M$, $\beta^{-1} * \alpha * \beta$ *is in M.*

Proof. Statement 1 follows from Lemma 2.20, Statement 2 from Lemmas 18 and 16, and Statement 3 from Lemma 15. ∎

THEOREM 2.2. *Suppose* $\mathscr{X} = \langle X, \geqslant, R_1, R_1, ... \rangle$ *is a linear structure. Then there exists a numerical structure* $\mathscr{Y} = \langle Re, \geqslant, S_1, S_2, ... \rangle$ *such that the following three statements are true*:

(1) $\mathscr{X}$ *and* $\mathscr{Y}$ *are isomorphic and all automorphisms of* $\mathscr{Y}$ *are of the form* $\alpha(x) = rx + s$, *where* $r > 0$ *and* $s \in Re$.

(2) *There exists a $\mathscr{Y}$-isomorphic representation of $\mathscr{X}$, and for all $\mathscr{Y}$-isomorphic representations φ and ψ of $\mathscr{X}$, there exist $r > 0$ and s in Re such that $\psi = r\varphi + s$.*

(3) *For each $r > 0$ and each s in Re, α defined by $\alpha(u) = ru + s$ for u in Re is an automorphism of $\mathscr{Y}$ and $r\varphi + s$ is $\mathscr{Y}$-isomorphic representation for $\mathscr{X}$ for each $\mathscr{Y}$-isomorphic representation φ of $\mathscr{X}$.*

Proof. Let A be the set of automorphisms of $\mathscr{X}$. Then $\langle X, \geqslant, A \rangle$ is a structure of linear automorphisms. Let M be the set of infinitesimal elements of A. Then by Theorem 2.1, $\langle M, * \rangle$ is commutative and M satisfies 1-point homogeneity and 1-point uniqueness. Thus $\mathscr{X}$ and M satisfy the hypotheses of Theorem 2.12 of Narens (1980), and therefore by that theorem let $\mathscr{N} = \langle Re^+, \geqslant, R_1', R_2',... \rangle$ be a numerical structure and φ be an $\mathscr{N}$-representation for $\mathscr{X}$ that is an isomorphism and such that for each α in M, $\varphi(\alpha)$ is an automorphism of $\mathscr{N}$ that is a multiplication by a positive real, and for each r in Re^+, multiplication r is $\varphi(\beta)$ for some β in M. Let $\mathscr{Y}$ be the log transformation of $\mathscr{N}$, i.e., $Re = \log(Re^+)$, $\geqslant = \log(\geqslant)$, $S_1 = \log(R_1)$, $S_2 = \log(R_2)$,..., and $\mathscr{Y} = \langle Re, \geqslant, S_1, S_2,... \rangle$.

(1) Since log is a one-to-one function, $\mathscr{Y}$ is isomorphic to $\mathscr{N}$ and therefore to $\mathscr{X}$. Let η be an isomorphism of $\mathscr{X}$ onto $\mathscr{Y}$. Let A' be the set of automorphisms of $\mathscr{N}$. Let $M' = \{\eta(\alpha) \mid \alpha \in M\}$ be the infinitesimal automorphisms of A'. Since multiplications by positive reals were the infinitesimal automorphisms of $\mathscr{N}$, and since $\mathscr{Y}$ is the log transformation of $\mathscr{N}$, M' consists of additions by reals, i.e., each α in M' is of the form $\alpha(x) = x + s$ for some s in Re, and all functions on Re of this form are in M'. Now let β be an arbitrary element of $A' - M'$. By Statement 3 of Theorem 2.1, $\beta * \alpha * \beta^{-1}$ is in M' for each α in M'. Thus for each α in M', let γ_α in M' be such that

$$\beta * \alpha * \beta^{-1} = \gamma_\alpha.$$

Then

$$\beta * \alpha = \gamma_\alpha * \beta. \tag{2.2}$$

Define f on Re as follows: for each α in M', if $\alpha(x) = x + r$, then $\gamma_\alpha(x) = x + f(r)$. Then by Eq. (2.2), for each x and r in Re,

$$\beta(x + r) = \beta(x) + f(r). \tag{2.3}$$

Putting $x = 0$ in Eq. (2.3), we get

$$\beta(r) = \beta(0) + f(r).$$

Thus

$$\beta(x + r) = \beta(x) + [\beta(r) - \beta(0)],$$

and therefore,

$$\beta(x + r) - \beta(0) = [\beta(x) - \beta(0)] + [\beta(r) - \beta(0)]. \tag{2.4}$$

Letting $g(u) = \beta(u) - \beta(0)$, Eq. (2.4) becomes

$$g(x + r) = g(x) + g(r). \tag{2.5}$$

Equation (2.5) is called *Cauchy's Equation*. Since β is an automorphism of $\mathscr{Y}$, it is order preserving, and thus g is strictly monotonic. It is a well-known theorem of analysis that a strictly monotonic g that satisfies Eq. (2.5) for all x, r in Re is of the form $g(u) = tu$ for some $t > 0$. Thus for some $t > 0$, $\beta(u) = tu + \beta(0)$ for all u in Re.

(2)   An isomorphism of $\mathscr{X}$ onto $\mathscr{Y}$ is a $\mathscr{Y}$-isomorphic representation for $\mathscr{X}$, and such an isomorphism exists by Statement 1. Suppose φ and ψ are $\mathscr{Y}$-isomorphic representations for $\mathscr{X}$. Then it is easy to verify that $\psi\varphi^{-1}$ is an automorphism of $\mathscr{Y}$, and thus by Statement 1 there exist $r > 0$ and s in Re such that for each t in Re,

$$\psi\varphi^{-1}(t) = rt + s,$$

and letting $x = \varphi^{-1}(t)$, it then follows that

$$\psi(x) = r\varphi(x) + s.$$

(3)   Let $r > 0$ and s be in Re.

Let α be defined by $\alpha(u) = ru + s$ for u in Re. We will show α is an automorphism of $\mathscr{Y}$. Let B_0 be the set of automorphisms of $\mathscr{Y}$ that are dilations at 0. By Statement 1, each β in B_0 is of the form $\beta(u) = tu + q$ for some $t > 0$ and $q \in Re$, which by $\beta(0) = 0$ reduces to $\beta(u) = tu$ for some t in Re. Thus each element of B_0 is a multiplication by a positive real. Since by Lemma 2.8, $\langle B_0, \geqslant, * \rangle$ is a Dedekind complete, dense, totally ordered group, it is not difficult to show using standard Dedekind completeness arguments about the positive reals that B_0 consists of all multiplications by positive reals. Thus in particular, β_r, defined by $\beta_r(u) = ru$ for u in Re, is in B_0. By construction, each infinitesimal element of $\mathscr{Y}$ is of the form $\gamma(u) = u + w$ where w is some element of Re. Let M' be the set of infinitesimal elements of $\mathscr{Y}$. Since by Lemma 20, $\langle M', \geqslant, * \rangle$ is a Dedekind complete, dense, totally ordered group, by the same line of reasoning as applied to $\langle B_0, \geqslant, * \rangle$ it follows that M' consists of all additions by reals, i.e., the function $\gamma(u) = u + q$ is in M' for each q in Re. In particular, α_s, defined by $\alpha_s(u) = u + s$, is in M'. Then α is an automorphism of $\mathscr{Y}$ since

$$\alpha(u) = ru + s = \alpha_s * \beta_r(u),$$

where $\alpha_s \in M'$ and $\beta_r \in B_0$.

Let φ be an arbitrary $\mathscr{Y}$-isomorphic representation for $\mathscr{X}$. By the above, let α be the automorphism of $\mathscr{Y}$ defined by $\alpha(u) = ru + s$ for u in Re. It is easy to verify that $\psi = \alpha\varphi$ is a $\mathscr{Y}$-isomorphic representation of $\mathscr{X}$ since both φ and α are structure

preserving. Since for each x in X, $\psi(x) = \alpha\varphi(x) = r\varphi(x) + s$, it follows that $\psi = r\varphi + s$. ∎

LEMMA 2.21. *Let $\langle Z, \succcurlyeq_r, D \rangle$ be a structure of linear automorphisms. Then each element of D is either a dilation or an infinitesimal.*

Proof. The proof of Statement 1 of Theorem 2.2 only used assumptions about the structure $\mathscr{X} = \langle X, \succcurlyeq, R_1, R_2,... \rangle$ that were identical $\langle X, \succcurlyeq, A \rangle$ being a structure of linear automorphisms where A is the set of automorphisms of $\mathscr{X}$. Thus by a proof that is almost identical to the proof of Statement 1 of Theorem 2.2, $\langle Z, \succcurlyeq_1, D \rangle$ is isomorphic to a structure of the form $\langle Re, \geqslant, D' \rangle$ where each automorphism α in D' is of the form $\alpha(x) = rx + s$ for some $r > 0$ and s in Re. Using this result, it is easy to verify that all automorphism in D' are either dilations or infinitesimals, which by isomorphism yields that all automorphisms in D are either dilations or infinitesimals. ∎

THEOREM 2.3. *There is no structure $\mathscr{Y} = \langle Y, \succcurlyeq, S_1, S_2,... \rangle$ such that the following three conditions simultaneously hold:*

(1) $\langle Y, \succcurlyeq \rangle$ *is of order type θ.*

(2) $\mathscr{Y}$ *satisfies 3-point homogeneity.*

(3) $\mathscr{Y}$ *satisfies 3-point uniqueness.*

Proof. Let a, b, d be elements of Y such that $a \prec b \prec d$. Let

$$H = \{\alpha \mid \alpha \text{ is an automorphism of } \mathscr{Y}\},$$

$$H_d = \{\alpha \mid \alpha \in H \text{ and } \alpha(d) = d\},$$

and

$$X = \{y \mid y \in Y \text{ and } y > d\}.$$

Let "$\alpha \upharpoonright X$" stand for the restriction of α to X, and let

$$A = \{\beta \mid \beta = \alpha \upharpoonright X \text{ for some } \alpha \text{ in } H_d\}.$$

Then $\langle X, \succcurlyeq, A \rangle$ is a structure of linear automorphisms. Let

$$M = \{\alpha \mid \alpha \text{ is an infinitesimal element of } A\}.$$

Then by Theorem 2.1, M satisfies 1-point uniqueness. Let

$$H_{a,d} = \{\alpha \mid \alpha \in H, \alpha(a) = a, \text{ and } \alpha(d) = d\},$$

$$H_{b,d} = \{\alpha \mid \alpha \in H, \alpha(b) = b, \text{ and } \alpha(d) = d\},$$

$$M_a = \{\beta \mid \beta = \alpha \upharpoonright X \text{ for some } \alpha \text{ in } H_{a,d}\},$$

and

$$M_b = \{\beta \mid \beta = \alpha \restriction X \text{ for some } \alpha \text{ in } H_{b,d}\}.$$

Then $M_a \subseteq A$ and $M_b \subseteq A$. Let γ be an arbitrary element of M_a and γ' in $H_{a,d}$ be such that $\gamma = \gamma' \restriction X$. If for some x in X, $\gamma(x) = x$, then $\gamma'(a) = a$, $\gamma'(b) = b$, and $\gamma'(x) = x$, which by 3-point uniqueness in $\mathscr{Y}$ yields $\gamma' = \iota$ and thus $\gamma = \iota$. Therefore the only dilation in A that is in M_a is the identity. Thus by Lemma 2.21, $M_a \subseteq M$. Similarly $M_b \subseteq M$. Let u, v be elements of X such that $u \prec v$. By 3-point homogeneity in $\mathscr{Y}$, let α', β' in H be such that

$$\alpha'(a) = a, \qquad \alpha'(d) = d, \qquad \alpha'(u) = v,$$

and

$$\beta'(b) = b, \qquad \beta'(d) = d, \qquad \beta'(u) = v,$$

and let $\alpha = \alpha' \restriction X$ and $\beta = \beta' \restriction X$. Then $\alpha \in M_a$ and $\beta \in M_b$, and thus α and β are in M. Since $\alpha(u) = \beta(u)$ and (by Theorem 2.1) M satisfies 1-point uniqueness, $\alpha = \beta$. Therefore $\alpha(v) = \beta(v)$. Thus $\alpha'(d) = \beta'(d)$, $\alpha'(u) = \beta'(u)$, and $\alpha'(v) = \beta'(v)$, which by 3-point uniqueness yields $\alpha' = \beta'$. Thus $\alpha'(b) = \beta'(b) = b$. Therefore $\alpha'(a) = a$, $\alpha'(b) = b$, and $\alpha'(d) = d$, which by 3-point uniqueness yields $\alpha' = \iota$. But this is impossible since $\alpha'(u) = v \succ u$. ∎

THEOREM 2.4. *Suppose* $\mathscr{X} = \langle X, \succcurlyeq, R_1, R_2, \ldots \rangle$ *is a monotonic structure. Then there exists* $\mathscr{N} = \langle Re^+, \geqslant, S_1, S_2, \ldots \rangle$ *such that the following two statements are true*:

(1)   *There exists an isomorphic* $\mathscr{N}$*-representation for* $\mathscr{X}$.

(2)   *The set of isomorphic* $\mathscr{N}$*-representations for* $\mathscr{X}$ *forms an ordinal scale, i.e.,* (i) *for each isomorphic* $\mathscr{N}$*-representation* φ *of* $\mathscr{X}$ *and each strictly increasing function* F *from* Re^+ *onto* Re^+, $F(\varphi)$ *is an isomorphic* $\mathscr{N}$*-representation of* $\mathscr{X}$, *and* (ii) *for all isomorphic* $\mathscr{N}$*-representations* φ *and* ψ *of* $\mathscr{X}$, *there exists a strictly increasing function* H *from* Re^+ *onto* Re^+ *such that* $\varphi = H(\psi)$.

Proof. (1) Since $\langle X, \succcurlyeq \rangle$ is of order types θ, by Theorem 1.1, let γ be an isomorphism of $\langle X, \succcurlyeq \rangle$ onto $\langle Re^+, \geqslant \rangle$. Let $\mathscr{N} = \langle Re^+, \geqslant, \gamma(R_1), \gamma(R_2), \ldots \rangle$. Then γ is an isomorphic $\mathscr{N}$-representation for $\mathscr{X}$.

(2) (i). Suppose φ is an isomorphic $\mathscr{N}$-representation of $\mathscr{X}$ and F is a strictly increasing function from Re^+ onto Re^+. For each x in X, let $\psi(x) = F[\varphi(x)]$. We will show that ψ is an isomorphic $\mathscr{N}$-representation for $\mathscr{X}$. It immediately follows that ψ is an isomorphism from $\langle X, \succcurlyeq \rangle$ onto $\langle Re^+, \geqslant \rangle$. Let $\alpha = \varphi^{-1}\psi$. Then it easily follows that α is an automorphism of $\langle X, \succcurlyeq \rangle$, We will now show that α is an automorphism of $\mathscr{X}$. Since $\mathscr{X}$ is a monotonic structure, $\langle X, \succcurlyeq \rangle$ is of order type θ. Thus by denumerable density, let Y be a subset of X of order type η such that for each u, v in X, if $u \succ v$ then for some y in Y such that $u \succ y \succ v$. Then, since α is strictly increasing, by η-homogeneity in $\mathscr{X}$, let β be an automorphism of $\mathscr{X}$ such that

$\beta(y) = \alpha(y)$ for all y in Y. Since β and α are both strictly increasing and agree on an order dense subset, Y, of $\langle X, \geqslant \rangle$, it is easy to show that $\beta = \alpha$. Thus $\alpha = \varphi^{-1}\psi$ is an automorphism of $\mathcal{X}$. Therefore, $\varphi\alpha = \varphi[\varphi^{-1}\psi] = \psi$ is an isomorphic $\mathcal{N}$-representation for $\mathcal{X}$. (ii). Now suppose φ and ψ are isomorphic $\mathcal{N}$-representations for $\mathcal{X}$. Define H on Re^{+} by: for each r in Re^{+}, $H(r) = \varphi(x)$ where x in X is such that $\psi(x) = r$. Then H is a strictly increasing function from Re^{+} onto Re^{+} such that $\varphi = H(\psi)$. ∎

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