

2 Intrinsic Archimedeaness and the Continuum

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1. INTRODUCTION

1.1 Background

The Archimedean axiom has its roots in ancient mathematics, where it was used to banish from consideration both infinitely large and infinitesimally small quantities. This was essentially the only rigorous means available to eliminate such quantities until very late in the nineteenth century when G. Cantor gave a fully rigorous description of the continuum in terms of an ordering relation. In a great many contexts, Cantor's method provides a different means for eliminating the infinitely large and small. Although the two approaches are quite different, they are interrelated in subtle ways. Part of this chapter focuses on such relationships.

Both the Archimedean and Cantorian approaches to the continuum use second-order logical concepts in their formulations. As will follow from results presented subsequently, some sort of higher-order logical concept is necessary in any description of the continuum. Thus, in particular, any part of science that uses the continuum necessarily assumes higher-order concepts.

Many who work in the foundations of science believe higher-order concepts to be inherently nonempirical and, thus, believe that scientific concepts based on a continuum include some nonempirical component. A similar situation exists for those scientific concepts that foundationally have bases in situations less rich than the continuum but nevertheless require imbeddability into structures based on the continuum. Usually an Archimedean axiom is used to effect the imbedding. Therefore, it is not surprising that each coherent, effective program for science to emerge has incorporated a theory of measurement based on some higher-order concept.

In practice, the Cantor axioms have not been widely used in measurement theory because of the nonconstructive nature of the axiom postulating the existence of a countable, order-dense subset (see below). Preference has been accorded the more constructive Archimedean approach, when it is available.

Historically, to assure that all magnitudes and differences of magnitudes are commensurable, the concept of Archimedeaness has been defined in terms of an operation, usually assumed to be associative. Its justification in these contexts has consisted in trying to make intuitively clear that, in terms of recursively generated applications of the operation as a method of determining size, no element is infinitely large with respect to another and that no two elements are infinitesimally close together. In this chapter, we seek to extend the concept of Archimedeaness—of commensurability—to general structures that may have no operation among its defining relations (*primitives*). In such situations, we see no way to keep Archimedeaness from becoming a much more abstract notion and correspondingly a much more difficult one to justify as correct.

Our approach is to formulate, in a very general fashion, what Archimedeaness should accomplish and then show that this imposes severe restrictions that are satisfied by only one concept (up to logical equivalence). In this approach, the resulting concept of Archimedeaness will be justified by theorems; intuition will play a role only at the beginning stages in stating what should be accomplished. Many of the theorems are difficult to prove and require concepts of abstract algebra, particularly those of group theory. No proofs will be presented in this paper, but references are provided to the original publications. Most of them can also be found in Luce, Krantz, Suppes, and Tversky (1990).

Our basic goal is to find the correct general concept and to justify it as such. Unfortunately our theory of Archimedeaness is not yet completely worked out; there are important gaps, which will be indicated throughout the chapter often as conjectures, tantalizing ideas, or unresolved technical questions.

1.2 The Research Agenda

Cantor (1895) gave the following simple and elegant characterization of the continuum: $C = \langle X, \succeq \rangle$ is said to be a *continuum*,¹ if and only if the following three conditions are met:

1. C is a totally ordered set without endpoints.
2. C is Dedekind complete (i.e., each nonempty bounded subset of X has a least upper bound in X).
3. There is a denumerable subset Y of X that is order dense in X (i.e., for each x, y in X , there exists z in Y such that $x > z > y$).

¹Throughout this chapter, we consider only continua without endpoints. The results easily generalize to situations with either one or two endpoints.

Cantor proved that each continuum is isomorphic to the ordered positive real numbers, which we designate as $\langle Re^+, \geq \rangle$.

Historically one of the principal uses made of Archimedean axioms was to establish the isomorphic imbedding of structures into ones based on a continuum. We take this to be a principal characteristic of Archimedeaness. The basic idea is as follows.

There are certain structures that shall be taken to be intrinsically Archimedean—we discuss what this might mean shortly. Such structures will take the following general form: $\langle X, \succeq, R_1, \dots, R_i, \dots \rangle$, where the R_i may be elements of X , relations on X , relations of relations of X , and so on, and $\langle X, \succeq \rangle$ is a continuum, and certain other conditions are satisfied that will be stated later. None of the R_i need be an operation or partial operation. A structure $\mathcal{S} = \langle S, \succeq, S_1, \dots, S_i, \dots \rangle$ is said to be *Archimedean* if and only if there is an intrinsically Archimedean structure \mathcal{X} and an isomorphism ϕ from \mathcal{S} into \mathcal{X} such that $\phi(S)$ is a dense subset of an open interval of \mathcal{X} . This definition consciously omits cases where the ordering on S may be discrete or have gaps in it. There are obviously discrete structures that are Archimedean (e.g., $\langle I^+, \geq, + \rangle$, where I^+ is the positive integers), and the approach presented in this chapter can be extended to such cases.

Given the general framework of distinguishing intrinsically Archimedean and Archimedean structures, the plan of the research is obvious: Provide a precise definition of intrinsically Archimedean, argue that it is the correct one, and then describe conditions that allow other structures to be appropriately imbedded into intrinsically Archimedean ones. Unfortunately this seems to be a difficult plan to carry out, and, as was mentioned, only partial results can be reported at this time. Mainly they concern attempts to capture the concept of intrinsic Archimedeaness. In fact, this chapter could well be entitled “Seeking the intrinsically Archimedean.”

2. POSITIVE CONCATENATION STRUCTURES

2.1 Archimedeaness in Standard Sequences

Archimedeaness has been traditionally defined in terms of operations, for which the following algebraic concepts are useful.

Let $\mathcal{X} = \langle X, \succeq, \circ \rangle$ be such that X is a nonempty set, \succeq is a binary relation on X , and \circ is a binary (closed) operation on X . Then \mathcal{X} is said to be a *concatenation structure*² if and only if the following four conditions are met:

²The general concept of a concatenation structure in the literature (e.g., Narens & Luce, 1976; Luce & Narens, 1985) assumes \circ is a partial, not a closed, operation and does not assume density. We are mainly concerned here with homogeneous structures, which are necessarily closed, and with the continuum, which is dense; thus, it is convenient to use the more restrictive definition.

1. X has at least two elements.
2. \succeq is a total ordering.
3. $\langle X, \succeq \rangle$ is dense.
4. \circ is (strictly) monotonic in each variable (i.e., for all x, y, z in X , $x \succeq y \leftrightarrow x \circ z \succeq y \circ z \leftrightarrow z \circ x \succeq z \circ y$).

The operation \circ is said to be

Positive if and only if $x \circ y > x$ and $x \circ y > y$ for all x, y in X .

Associative if and only if $x \circ (y \circ z) = (x \circ y) \circ z$ for all x, y, z in X .

Commutative if and only if $x \circ y = y \circ x$ for all x, y in X .

Idempotent if and only if $x \circ x = x$ for all x in X .

Further, \mathcal{X} is said to be *Archimedean in standard sequences*, often written *ss-Archimedean*, if and only if for each x and y in X , there exists a positive integer n such that $nx > y$, where the *standard sequence* $\{kx\}$ is defined inductively as follows: $1x = x$, and for $k > 1$, $kx = [(k - 1)x] \circ x$.

Note that, for positive structures, standard sequences are increasing, and so *ss-Archimedeanness* can be used to say there are no infinitely large elements. However, for idempotent structures, *ss-Archimedeanness* is false, because $kx = x$ for all positive integers k . Other forms of the Archimedean axiom will be given later that apply to both positive and nonpositive structures.

2.2 A Problematic Example

Example 2 below will show that *ss-Archimedeanness* is problematic even for positive operations, but first we present a nonproblematic example.

EXAMPLE 1. Let $\mathcal{E}_1 = \langle \mathcal{R}e^+, \succeq, + \rangle$. \mathcal{E}_1 is defined on a continuum and is positive, associative, and commutative. \mathcal{E}_1 is an archetypical example of an Archimedean structure, and if any structure is going to be described as intrinsically Archimedean, then \mathcal{E}_1 certainly will. It is interesting to ask if there are any other intrinsic Archimedean structures, not isomorphic to \mathcal{E}_1 , that are positive, associative, and commutative. The following example is very instructive.

EXAMPLE 2. Let β denote some object that is not a real number, and let \mathcal{R} denote the set of nonnegative real numbers. Let (x_0, \dots, x_i, \dots) denote the infinite sequence whose i th term is x_i . Let B be the set of all sequences each of whose terms, except for one, is the element β ; the exceptional one is an element of \mathcal{R} . The notation $\vec{a}_i = (\beta, \dots, a_i, \beta, \dots)$ indicates both that the element of \mathcal{R} is in the i th term of the sequence \vec{a}_i and that that term has the numerical value a_i . Let $A = B - \{0, \beta, \beta, \dots\}$. Suppose \vec{a}_i, \vec{b}_j , and \vec{c}_k are arbitrary elements of A . Define \succeq and \circ as follows:

$$\vec{a}_i \succeq \vec{b}_j \text{ iff either } i = j \text{ and } a_i \geq b_i \text{ or } i > j,$$

and

$$\vec{a}_i \circ \vec{b}_j = \vec{c}_k \text{ iff } i + j = k \text{ and } a_i + b_j = c_k.$$

Let $\mathcal{E}_2 = \langle A, \succeq, \circ \rangle$. Then \mathcal{E}_2 is a concatenation structure; $\langle A, \succeq \rangle$ is a continuum; and \mathcal{E}_2 is positive, associative, commutative, and *ss*-Archimedean. It is easy to show that \mathcal{E}_2 is not isomorphically imbeddable in $\langle \mathbb{R}e^+, \succeq, + \rangle$. Although \mathcal{E}_2 satisfies *ss*-Archimedeaness, do we really want to call it Archimedean? Clearly, because of *ss*-Archimedeaness, it does not have one element infinitely larger than another (in terms of \circ). However, the elements of the sets

$$A_i = \{\vec{a}_i \mid \vec{a}_i \in A \text{ and } a_i \in \mathbb{R}\}$$

look like they are infinitesimally close to one another.

2.3 Archimedeaness in Standard Differences

However, how are we going to define what it means for \vec{a}_i and \vec{b}_j to be infinitesimally close in a context like Example 2? Given generic elements a and b , the usual way is to find an element c that represents their difference, (e.g., if $a > b$, then $a = b \circ c$) and show that some element of A is infinitely large with respect to c . However, for elements of A_i , no such c can be found, so this strategy is not immediately applicable. Another approach is to try to use an alternative axiom that Roberts and Luce (1968) proposed for extensive measurement structures for which solvability was not a postulated condition. It is this: A structure is said to be *Archimedean in standard differences*, written *sd*-Archimedean, if and only if, for all x, y, u, v in A , if $x > y$, then there exists a positive integer n such that

$$nx \circ u \succeq ny \circ v.$$

This axiom was motivated in Krantz, Luce, Suppes, and Tversky (1971, p. 74) as follows (with the notation changed to ours):

It should be noted that [the *sd*-Archimedean axiom] is, in fact, the ordinary Archimedean property for differences. For, if we define $(x, y) \succeq (u, v)$ to mean $x \circ u \succeq y \circ v$, then [the *sd*-Archimedean axiom] simply says that if (x, y) is positive (i.e., $x > y$) then for some positive integer n ,

$$n(x, y) = (nx, ny) \succeq (u, v).$$

The problem with their motivation is in justifying the last equation. The inequality $(nx, ny) \succeq (u, v)$ does seem to capture adequately the idea that the difference between n copies of x and n copies of y is greater than or equal to the difference between u and v . However, this in itself is not sufficient to say that the difference between x and y is not infinitely close. To do that, one needs the equation $n(x, y) = (nx, ny)$ which identifies n copies of the difference between x and y with the difference between n copies of x and n copies of y . However, this latter equation is not justified; it is simply taken as a definition of what n copies of the difference means.

Let us look at this problem in another way. Suppose $x > y \circ z$. Then it is reasonable to say that the “difference between x and y is greater than z .” Using this concept of difference, we can formulate the idea of x and y not being infinitely close by requiring that, for each element w , a positive integer n can always be found for which n copies of z exceeds w . This approach, which relies entirely on being able to find a z such that $x > y \circ z$, fails when no such z exists, as is the case for elements from A_i .

An obvious strategy is to attempt to extend (i.e., imbed) the structure \mathcal{E}_2 to (in) a structure $\mathcal{E}'_2 = \langle A', \succeq', \circ' \rangle$ that allows one to find a z such that $x > y \circ' z$ whenever $x > y$. When x and y belong to A_i , this z will belong to $A' - A$ and will be infinitesimally small with respect to elements of A (i.e., if w is in A , then $w > nz$, for all positive integers n). As an example, let $A' = A \cup Re^+$. Extend \succeq to \succeq' by requiring all elements of A to be $>'$ all elements of Re^+ , and by requiring \succeq' restricted to Re^+ to be the usual ordering \succeq on Re^+ . Extend \circ to \circ' as follows: For all r and s in Re^+ ; and all \vec{a}_i and \vec{b}_j in A ,

$$r \circ' s = r + s,$$

and

$$r \circ' \vec{a}_i = \vec{a}_i \circ' r = \vec{b}_j \text{ iff } i = j \text{ and } a_i + r = b_j.$$

Then Re^+ are the infinitesimals, and, for all x and y in A' , if $x >' y$, then, for some z in A' , $x >' y \circ' z$. \mathcal{E}'_2 is a concatenation structure that is positive, associative, commutative, but it is not ss -Archimedean. If we let (x, y) stand for the difference between x and y , then, when $x = y \circ' z$, $z = (x, y)$, and, thus,

$$nz = n(x, y),$$

and by using commutativity and associativity, it follows that

$$nx = n(y \circ' z) = ny \circ' nz,$$

and, thus,

$$nz = (nx, ny).$$

However, does the previous argument really justify assuming the sd -Archimedean axiom? After all, one might just as well find a way of assigning infinitesimals to appropriate differences in an extended structure that is not associative and not commutative so that the previous argument will not go through. The sd -axiom might be easier to justify by observing that, in \mathcal{E}_2 , functions of the form $\gamma_n(x) = nx$ preserve the relations \succeq and \circ , that is for all x, y in A ,

$$x \succeq y \text{ iff } \gamma_n(x) \succeq \gamma_n(y),$$

and

$$\gamma_n(x \circ y) = \gamma_n(x) \circ \gamma_n(y).$$

Perhaps we should require, as is often done in considerations of meaningfulness, that structure-preserving maps extend to structure-preserving maps in the extension (see Chapter 22 of Luce et al., 1990). If we do so in the case of extensions of \mathcal{E}_2 that introduce infinitesimal differences, one retains $n(x, y) = (nx, ny)$ as before. This latter approach also applies to situations where the original operation \circ need be neither associative nor commutative. It needs only to satisfy $\gamma_n(x \circ y) = \gamma_n(x) \circ \gamma_n(y)$, a condition that often obtains, as we shall see.

2.4 Associative Structures on the Continuum

The following theorem can be shown:

THEOREM 2.1. *Let $\mathcal{X} = \langle X, \succeq, \circ \rangle$ be a concatenation structure, $\langle X, \succeq \rangle$ be a continuum, and \mathcal{X} be positive and associative. Then the following five statements are equivalent:*

1. \mathcal{X} is isomorphic to $\langle \mathbb{R}e^+, \succeq, + \rangle$.
2. \mathcal{X} satisfies the *sd*-Archimedean axiom.
3. \mathcal{X} satisfies the *ss*-Archimedean axiom and right restricted solvability (i.e., for all x, y in X , if $x > y$, then, for some z in X , $x > y \circ z$).
4. \mathcal{X} is right solvable in the sense that, for all x, y in X , if $x > y$, then, for some z in X , $x = y \circ z$.
5. \mathcal{X} is homogeneous in the sense that for each x and y in \mathcal{X} , an automorphism³ β of \mathcal{X} exists such that $\beta(x) = y$.

This theorem and the previous discussion somewhat justify the following assertion: *All intrinsically Archimedean concatenation structures that are positive and associative are isomorphic to $\langle \mathbb{R}e^+, \succeq, + \rangle$ (and are, therefore, also commutative).*

The subtle interplay between the concepts of *ss*-Archimedeanness, right-restricted solvability, right solvability, homogeneity, and Dedekind completeness, such as is described in Theorem 2.1 for the special case of a positive and associative operation, will be explored more generally throughout this chapter. Homogeneity will play an especially key role in our investigations.

It is nearly trivial to see that statement 1 implies each of the others. Proofs of the converses were, we believe, first given in the following sources: Statement 2 in Roberts and Luce (1968; see Theorem 3.1 in Krantz et al., 1971); statement 3 in Krantz et al. (1971, Theorem 3.3); statement 4 is a variant on a result for ordered groups due to Loonstra (1946; see Fuchs, 1963, p. 47; it also follows from Theorem 2.1 of Luce and Narens, 1985, which shows, in a far more general

³An *automorphism* of a structure \mathcal{X} is an isomorphism from \mathcal{X} onto itself.

setting, that statement 4 implies *ss*-Archimedeaness); statement 5 is an unpublished result of Michael Cohen (1986; among other things, it shows that, in certain instances, homogeneity implies right-restricted solvability).

2.5 Additional Problematic Examples

There are certain inherent difficulties in trying to classify more general structures as being Archimedean, as the following two examples make clear:

EXAMPLES 3 and 4. Let \oplus be the following operation on Re^+ :

$$\begin{aligned} x \oplus y &= (x + y)/(1 + xy), \text{ if } x \text{ and } y < 1; \\ x \oplus y &= x + y, \text{ otherwise.} \end{aligned}$$

(Note that \oplus , when restricted to the positive reals < 1 , is the relativistic “addition” formula for velocities *less than* the velocity of light—1 in this representation.) Let

$$\mathcal{E}_3 = \langle Re^+, \geq, \oplus \rangle.$$

It is easy to verify that \oplus is positive, commutative, and right restrictively solvable. It is not, however, *ss*-Archimedean, because n copies of $1/2$ is < 1 for all positive integers n . Let

$$\mathcal{E}_4 = \langle Re^+, \geq, \oplus, + \rangle.$$

\mathcal{E}_3 and \mathcal{E}_4 are very closely related: \mathcal{E}_4 is definable from the primitives \oplus and \geq of \mathcal{E}_3 as follows:⁴

$$\begin{aligned} x + y = z \text{ iff } \exists w \forall u \forall v [(u < w \ \& \ v < w \rightarrow u \oplus v < w), \text{ and} \\ (z \oplus w) \oplus w = (x \oplus w) \oplus (y \oplus w)]. \end{aligned}$$

The structures \mathcal{E}_3 and \mathcal{E}_4 describe the same situation, \mathcal{E}_4 in a little more redundant way. Thus, if one is to be Archimedean or nonArchimedean, then the other should be the same. Observe, however, that, in terms of \oplus , \mathcal{E}_4 appears not to be Archimedean, whereas in terms of $+$, it does appear to be Archimedean. At this point, it is best not to call it either but to investigate further what the consequences might be in making such distinctions in general.

2.6 Archimedeaness in Difference Sequence

The simplest (and oldest) example of a qualitative concatenation structure is the classical model for additive physical quantities, called *extensive*, which was

⁴In this definition, w is forced to have the value 1, which acts like the velocity of light in the relativistic “addition” formula; however, by the way \oplus was defined, it follows that, for all $x < 1$, $x \oplus 1 \neq 1$ rather than $= 1$, which would follow from the relativistic formula. Furthermore, because 1 is not among the primitives of \mathcal{E}_3 , we cannot explicitly mention it in the definition.

described in Theorem 2.1. There the concept of *ss*-Archimedeaness is a direct analogue of the definition used in the real number system. Note that, because the structure is associative and commutative, it does not really matter how we compose n copies of x , because, for example, $((x \circ x) \circ x) \circ x$, $(x \circ x) \circ (x \circ x)$, and $x \circ (x \circ (x \circ x))$ are all equivalent.

When it comes to working with general, nonassociative concatenation structures, which are the ones of interest when any form of averaging is involved or when the variables have a factorial structure, as is often true in the behavioral and social sciences, there are at least two possible sources of trouble in defining Archimedeaness.

The first is that, in the positive case which is the natural generalization of extensive structures, there are an indefinite number of distinct ways of defining an infinite standard sequence, and none seems outstandingly better than the others. The one commonly selected is the inductive definition given earlier, and, in one case of major interest (positive, right restrictively solvable, and homogeneous), the choice is immaterial, because it implies the same property for all definitions of standard sequences.⁵ However, we have no proof that this is so in a completely general positive concatenation structure. This is an important open problem.

The following example, due to Margaret Cozzens (personal communication), shows that the definition of a standard sequence definitely matters in some nonidempotent concatenation structures.

EXAMPLE 5. Let $\mathcal{E}_5 = \langle Re^+, \geq, \oplus \rangle$, where, for x, y, z in Re^+ , $x \oplus y = z$ iff $x + \frac{1}{2}y = z$.

By induction, it is not difficult to show that $nx = x(n + 1)/2$, but for x_n defined by $x_1 = x$ and $x_n = x \oplus x_{n-1}$, then, by induction, $x_n = x(2^n - 1)/2^{n-1} \leq 2x$. Thus, it is Archimedean in nx but not in x_n .

The other trouble arises when we turn to the class of idempotent structures, typified by averaging, that are necessarily nonassociative but also are not positive. For such structures, standard sequences according to any definition of repeated combinations of an element with itself are all trivial, and so, in particular, the definitions of *ss*- and *sd*-Archimedeaness are useless. An alternative adopted for such cases is to say that $\{x_n\}$ is a *difference sequence* iff there exist u, v in X with $v > u$ such that, for each n , $x_n \circ u = x_{n-1} \circ v$. Then the structure is said to be *Archimedean in difference sequences*, abbreviated *ds*-Archimedean, if and only if each bounded difference sequence is finite.

⁵The proof of this follows from the methods of proof used in Lemmas 2.4.8 and 2.4.9 of Narens (1985) or from similar theorems of Cohen and Narens (1979), which show how to translate Archimedean-like conditions of the structure to its automorphism group and vice versa.

Note that this third concept of Archimedeaness is defined for all concatenation structures, including positive ones. Thus, one would like to know, for various general classes of positive structures, the implication relationships among the three Archimedean axioms. Observe that the standard sequence definition really has bite only if the operation is quite generally defined, whereas the difference sequence definition has bite only if the structure is *solvable* in the sense that, given $x, y \in X$ with $x < y$, there exist z, z' in X such that $x \circ z = z' \circ x = y$.

2.7 Relations Among Three Concepts of Archimedeaness

Two additional definitions are needed to formulate what is known about the relations among the three kinds of Archimedean axioms thus far considered: $\mathcal{X} = \langle X, \succeq, \circ \rangle$ is said to be *uniformly, restrictedly right solvable* if and only if, for each x, y, r, s in X with $x < y$ and $r < s$, there exists z in X such that for all u in X for which $r \preceq u \preceq s$, $(u \circ x) \circ z \preceq u \circ y$. The n -copy operator nx (n a positive integer) is said to be *operation preserving* if and only if, for all x, y in X , $n(x \circ y) = nx \circ ny$.

THEOREM 2.2 (Luce, 1987, Theorem 3.3) *Suppose \mathcal{X} is a positive concatenation structure.*

- (i). *If \mathcal{X} is uniformly, restrictedly right solvable, then ss -Archimedeaness implies ds -Archimedeaness.*
- (ii). *If \mathcal{X} is right solvable, then ds -Archimedeaness implies ss -Archimedeaness.*
- (iii). *Suppose \mathcal{X} is restrictedly right solvable and the n -copy operators are operation preserving for each positive integer n . Then the following is true:*
 - (a) *ss -Archimedeaness implies sd -Archimedeaness;*
 - and*
 - (b) *if \mathcal{X} is order dense, then sd -Archimedeaness implies ss -Archimedeaness.*

It is worth noting that \mathcal{E}_2 (Example 2) is ss -Archimedean but not ds -Archimedean.

The next result establishes that, on the continuum, solvability is sufficient to get ss - and ds -Archimedeaness.

THEOREM 2.3 (Luce & Narens, 1985, Theorem 2.1) *Suppose \mathcal{X} is a Dedekind complete concatenation structure. If \mathcal{X} is positive and right solvable, then \mathcal{X} is ss -Archimedean. If \mathcal{X} is (right and left) solvable, then \mathcal{X} is ds -Archimedean.*

3. CLASSIFICATION OF AUTOMORPHISM GROUPS

3.1 Automorphism Groups of PCSs

One of the more striking discoveries about concatenation structures that are positive, *ss*-Archimedean, and restrictedly right solvable—the so-called PCSs—is that the Archimedeaness of the structure devolves to the important algebraic structure called the automorphism group, where an *automorphism* is an isomorphism of a structure with itself. (Physicists call automorphisms “symmetries” of the structure.) Under function composition, the set of all automorphisms forms a mathematical group (closed and associative operation, an identity, and inverses). Moreover, a partial order can be imposed on the group in terms of the asymptotic behavior of the automorphisms as follows: if α and β are automorphisms, then $\alpha \succeq' \beta$ if and only if there exists some x in X such that, for all $y > x$, $\alpha(y) \succeq \beta(y)$. The relation \succeq' is called the *asymptotic ordering* of the automorphism group. For PCSs, one can in fact show that \succeq' is connected and much more:

THEOREM 3.1 (Cohen & Narens, 1979, Theorem 2.3) *If \mathcal{X} is a PCS, then its automorphism group together with its asymptotic ordering is an Archimedean ordered group.*

Thus, by Hölder’s (1901) theorem, the automorphism group is isomorphic to a subgroup of the additive reals, and, therefore, it has a very simple structure, including being commutative. This comes as somewhat of a surprise, because PCSs are moderately weak structures when compared to the ordered structures generally encountered in algebra. Of course the most irregular of them have no automorphisms aside from the identity. In addition, some, for example,

$$x \circ y = x + y + (xy)^2,$$

which from many points of view are highly regular, also have no nontrivial automorphisms.

One can view a great deal of the recent work in measurement theory as exhibiting two major thrusts. The one that we are reporting on here attempts to gain a deeper understanding of the conditions for which some version of Theorem 3.1 holds for general relational structures other than PCSs. Although much is now known about this, there are still major gaps to be filled in. For example, we still cannot say anything general about the automorphism groups of arbitrary idempotent structures. Only by imposing additional restrictions do we get results of interest (see Sections 3.2, 4.1, and 4.6), both for concatenation structures and for much more general ones.

The other approach is based on the fact that a good deal is known about the possible automorphism groups for structures defined on the continuum (see

Theorem 6.4). The strategy followed is to accept some fairly weak structural assumptions (e.g., those of a concatenation structure) and then attempt to characterize the resulting kinds of structures in terms of the possible kinds of automorphism groups. Such a strategy is effective because the automorphism groups are highly constrained, as is formulated in Theorem 6.4. This approach has been carried out to some extent for concatenation (see Theorem 4.3) and conjoint structures (Luce & Narens, 1985).

3.2 Dilations, Translations, Homogeneity, and Uniqueness

$\mathcal{X} = \langle X, \succeq, S_j \rangle_{j \in J}$ is said to be an *ordered relational structure* if X is a set, \succeq is a weak ordering of X (that is, a transitive and connected relation on X), J is a set of integers, and, for each $j \in J$, S_j is a relation on X of finite order. As was noted above, an automorphism of \mathcal{X} is any isomorphism of the structure with itself.

The most important classifications of automorphisms of a continuum so far discovered are of three main types. The first is the distinction between *dilations*, which are automorphisms with at least one fixed point (i.e., an element a in X such that $\alpha(a) = a$), and the *translations*, which are those with no fixed point. It is convenient to treat the identity map as both a dilation and a degenerate translation. In terms of the familiar (nonnegative) affine transformations of the real numbers, the dilations are of the form $\alpha(x) = rx + s$, where r is positive and $\neq 1$, or $r = 1$, and $s = 0$, and the translations are of the form $\alpha(x) = x + s$. The major distinction between the two types of transformations is that, in terms of the asymptotic ordering, the translations are all infinitesimal relative to each dilation with $r > 1$. Furthermore, in terms of the asymptotic order and function composition, the translations with $s > 0$ form a solvable *ss*-Archimedean extensive structure (or, put in algebraic terms, the entire set of translations form an *Archimedean ordered group*). In the case of PCSs, all automorphisms are translations.

The second distinction has to do with the scope of the action of the automorphism group or, put another way, with the extent to which symmetries exist. A subset \mathcal{H} of automorphisms is said to be *M-point homogeneous* if and only if for each pair of similarly strictly ordered M -tuples of elements, there is an automorphism that maps the one M -tuple into the other. If the entire automorphism group is *M-point homogeneous*, then the structure itself is said to be *M-point homogeneous*. A structure that is *M-point homogeneous* for some $M \geq 1$ is called *homogeneous*. In terms of the nonnegative affine transformations as the automorphism group of some structure, the translations are 1-point homogeneous, and the group itself is 2-point homogeneous.

The third distinction has to do with the level of redundancy in the automorphism group. A subset \mathcal{H} of automorphisms is said to be *N-point unique* if and only if whenever two automorphisms agree at N or more distinct points, then they agree everywhere. A structure is said to be *unique* if its automorphism group

is N -point unique for some finite N . In terms of the nonnegative affine group, the translations are 1-point unique and the entire group is 2-point unique.

If the largest degree of homogeneity of a structure is M and the least degree of uniqueness is N , then the automorphism group is said to be of *scale type* (M, N) . Thus, a homogeneous PCS, which can be shown to be isomorphic to a real PCS with all translations as its automorphism group (Cohen and Narens, 1979), is of scale type $(1, 1)$. Any structure with the entire nonnegative affine group as its automorphism group is of scale type $(2, 2)$.

A major question is to understand fully the types of measurement structures that can arise in the sense of classifying the possible automorphism groups and then developing a description of classes of structures having those groups. This has been completed for all homogeneous and unique structures on the positive real numbers (Theorem 6.4) and is partially done for more general structures in which the translations form, as they do in the case just mentioned, a group that is both $(1, 1)$ and Archimedean ordered. We first look into these questions for concatenation structures and then move on to more general ones.

4. HOMOGENEOUS, UNIQUE CONCATENATION STRUCTURES

4.1 Two General Results

Our first result classifies both the structures and scale types that are possible for homogeneous and unique concatenation structures.

THEOREM 4.1 (Luce & Narens, 1985, Theorem 2.2) *Suppose \mathcal{X} is a homogeneous concatenation structure. Then \mathcal{X} is either idempotent, weakly positive (for all x , $x \circ x > x$), or weakly negative (for all x , $x \circ x < x$). If \mathcal{X} is also unique, then either $N = 1$ or both $N = 2$ and \mathcal{X} is idempotent.*

We see, therefore, that, for homogeneous and unique concatenation structures, the possible scale types are just $(1, 1)$, $(1, 2)$, and $(2, 2)$. At this level of generality, aside from the PCS case (Theorem 3.1), we know very little about the structure of the automorphism group. In particular, we would like to know when the translations form an Archimedean ordered group. One reason for interest in this question is the following:

THEOREM 4.2 (Luce, 1987, Theorem 3.5) *Suppose \mathcal{X} is a concatenation structure for which the set of translations forms a homogeneous, Archimedean-ordered group. Then the structure is ds-Archimedean and, if it is positive, it is also ss-Archimedean.*

As we shall see in Section 6, under these conditions, any ordered relational structure has a numerical representation of a particular type.

Before turning to such structures, some comment needs to be made about what it means to assume the translations form an ordered group. The group property can easily be shown to be equivalent to assuming the translations are 1–point unique (Luce, 1986, Theorem 2.1). The assumption that the asymptotic order is a total order is tantamount to assuming that the translations do not *cross* in the sense that there are x and y such that $\gamma(x) > x$ and $\gamma(y) < y$. (In the order-dense, Dedekind complete case, such crossing is impossible because it implies the automorphism has a fixed point.)

4.2 Unit Concatenation Structures

In the case of a homogeneous, unique concatenation structure on the positive reals—the Dedekind complete case—the operation has a particularly simple form:

THEOREM 4.3 (Cohen & Narens, 1979; Luce & Narens, 1985, Theorems 3.9, 3.12, and 3.13) *Suppose $\mathcal{R} = \langle Re^+, \geq, \odot \rangle$ is a concatenation structure that is homogeneous and unique. Then there is a function f from Re^+ onto Re^+ that is strictly increasing, $f(x)/x$ is strictly decreasing, and the operation \odot defined by $x \odot y = yf(x/y)$ is such that \mathcal{R} is isomorphic to $\langle Re^+, \geq, \odot \rangle$.*

Cohen and Narens (1979) called this kind of representation a *unit* (concatenation) *structure*. (They dealt only with the positive case; however, their methods and concepts extend to the general case, and this is expressed in Theorem 4.3.) The translations are simply multiplication by positive constants, and so they form an Archimedean-ordered group and Theorem 4.2 applies. It is easy to verify that the n –copy operators are of the form $nx = xf^{n-1}(1)$, and, thus, each is an automorphism. In particular, $n(x \odot y) = nx \odot ny$. Section 6 will generalize the concept of unit structures to general ordered relational structures.

For this class of structures, the following is true:

THEOREM 4.4 (Luce, 1987, Corollary to Theorem 3.3) *Suppose \mathcal{X} is a positive (homogeneous) unit concatenation structure. Then, in addition to being ds – and ss –Archimedean, it is sd –Archimedean.*

5. INTRINSIC ARCHIMEDEANNESS: A POSSIBLE DEFINITION

Examples 3 and 4, and even more pathological ones that can be easily devised, quickly lead one to the following conclusion: *In an intrinsically Archimedean structure, every reasonable positive operation that is definable from the primitives should be sd –Archimedean.* Of course, for this assertion to be effective,

“reasonable” and “definable” need to be given precise definitions—which will be done shortly.

First, we consider what we might mean by reasonable. As Example 2 shows, ss -Archimedeaness and positivity by themselves are not effective in eliminating infinitesimally close elements in a concatenation structure. Therefore, some additional or stronger conditions are needed. Theorem 2.3 suggests that right-restricted solvability might suffice. However, it is, in general, too strong a condition because there are structures with positive operations that are clearly Archimedean and are not right restrictedly solvable. As a case in point, consider the following:

EXAMPLE 6. $\mathcal{E}_6 = \langle Re^+, \geq, \oplus' \rangle$, where, for all x, y, z in Re^+ , $x \oplus' y = z$ if and only if $2x + 2y = z$. In this structure, $1.5 > 1$, but $1 \oplus' w > 1.5$ for all w in Re^+ .

The condition we shall focus upon in this chapter for capturing intrinsic Archimedeaness is homogeneity. At present, we do not have an adequate theory of intrinsic Archimedeaness for nonhomogeneous cases. For homogeneous structures with positive operations \circ on the continuum, we know, by the remarks following Theorem 4.3, that the equation $n(x \circ y) = nx \circ ny$ is valid for all elements x and y of the domain and all positive integers n . Furthermore, because of this and the earlier discussion of the sd -Archimedean axioms, we feel somewhat confident about invoking the sd -Archimedean axiom as a necessary condition for intrinsic Archimedeaness in homogeneous situations with positive operations. (Observe that \mathcal{E}_6 above is homogeneous and sd -Archimedean.) With these considerations in mind, we will, for homogeneous structures, adopt sd -Archimedeaness of definable positive operations as a critical characteristic of intrinsic Archimedeaness. However, it should be noted that, in some circumstances, this requirement is empty because there may be no positive operation definable from the primitive relations that make up the structure.

Second, we consider what we might mean by definable. Although there is no agreed upon general definition of what it means for a relation—in particular, an operation—to be definable in terms of given relations, for a number of specific concepts, it can be shown that the defined relation must be invariant under the automorphisms of the given structure, and it is widely agreed that any general definition should exhibit this property. Of course there may be invariant relations that are not definable, e.g., the relation may only exist through the Axiom of Choice. We make the invariance condition explicit.

Let \mathcal{X} and \mathcal{C} be structures that have a common domain D . Then \mathcal{C} is said to be *invariant under the automorphisms of \mathcal{X}* if and only if, for each automorphism β of \mathcal{X} , each n -ary primitive relation R of \mathcal{C} and each a_1, \dots, a_n in D , the following is satisfied:

$$R(a_1, \dots, a_n) \text{ iff } R[\beta(a_1), \dots, \beta(a_n)].$$

If the primitives of \mathcal{C} are defined from the primitives of \mathcal{X} through first-order, second-order, or any higher-order logic, then it can be proved that \mathcal{C} is invariant under the automorphisms of \mathcal{X} . (For a detailed description of the relationships between definability concepts and invariance, see Narens, 1988.) Thus, the structures with domain D that are “definable” from \mathcal{X} are a subset of those that are invariant under the automorphisms of \mathcal{X} . Further, because of this inclusion, invariance under automorphisms is a good generalization of first-order, second-order, and so forth definability.

A structure $\mathcal{X} = \langle X, \succeq, R_1, \dots \rangle$ is said to be *intrinsically z -Archimedean*, where $z = ss, sd, \text{ or } ds$, if and only if the following hold:

1. $\langle X, \succeq \rangle$ is a continuum.
2. There exists a (positive) operation on X that is z -Archimedean and invariant under the automorphisms of \mathcal{X} .
3. Any other (positive) operation on X that is invariant under the automorphisms of \mathcal{X} is also z -Archimedean.

The following striking theorem can be shown.

THEOREM 5.1 *Suppose $\mathcal{X} = \langle X, \succeq, R_1, \dots \rangle$, the automorphisms of \mathcal{X} form a homogeneous, Archimedean-ordered group, and $\langle X, \succeq \rangle$ is a continuum. Then \mathcal{X} is intrinsically ss -, sd -, and ds -Archimedean.*

Theorem 5.1 is an immediate consequence of Theorems 4.2, 4.3, and 4.4.

Although Theorem 5.1 covers some important cases, others are not covered. First, there are weakly positive structures that fail to be positive, so they are not covered. Some, such as Example 5, are decidedly ambiguous as to ss -Archimedeaness. Second, there are homogeneous idempotent structures for which no positive concatenation structure is automorphism invariant because they are of scale type $(1, 2)$ or $(2, 2)$. Some of these structures are remarkably Archimedean, as, for example, $\langle Re, \succeq, \circ \rangle$, where $x \circ y = \frac{1}{2}(x + y)$. For these, we can at best expect to arrive at ds -Archimedeaness. Toward that end, the next section investigates the consequences of a relational structure having a homogeneous, Archimedean ordered group of translations.

6. HOMOGENEOUS, ORDERED RELATIONAL STRUCTURES

The main result of this section characterizes, in terms of a particularly nice kind of numerical representation, those general, ordered relational structures whose translations form a homogeneous, Archimedean ordered group. As we saw in Theorem 4.2, these conditions on the translations of a concatenation structure

imply ds -Archimedeaness and, as will be seen shortly, lead to a numerical representation (which is one of the major reasons for invoking Archimedeaness); therefore, it is possible that, for homogeneous structures, the property of the translations forming an Archimedean ordered group is a suitable generalization of Archimedean operations. An ultimate decision on this awaits a deeper understanding of Archimedeaness at the structural level when no operation is present as a primitive, which is partially clarified in Theorem 6.5.

6.1 Real Unit Structures

To formulate the main result of the section, we need to give a generalization of the concept of a real unit structure, introduced for PCSs by Cohen and Narens (1979) and later generalized to all concatenation structures on Re^+ by Luce and Narens (1985) (see Theorem 4.3 of Section 4.2). We continue to use the same term in the general case.

Suppose $\mathcal{R} = \langle R, \geq, R_j \rangle_{j \in J}$, where $R \subseteq Re^+$, is a numerical relational structure. \mathcal{R} is said to be a *real unit structure* if and only if there exists $T \subseteq Re^+$ such that the following conditions are met:

1. T is a group under multiplication.
2. T maps R into R .
3. T restricted to R is the set of translations of \mathcal{R} .

THEOREM 6.1 [Luce, 1987, Theorem 5.1.(i) and (ii)] *Suppose \mathcal{X} is an ordered relational structure. Then \mathcal{X} is isomorphic to a real unit structure with a homogeneous group of translations if and only if the translations of \mathcal{X} together with the asymptotic ordering form a homogeneous, Archimedean-ordered group.*

COROLLARY. *If, in addition, \mathcal{X} is order dense, then the automorphism group of its unit representation is a subgroup of the nonnegative affine group restricted to R .*

As we saw in Theorem 4.3, real unit concatenation structures have a particularly nice form.

A further equivalence to the translations forming a homogeneous, Archimedean ordered group, one that is of great relevance to dimensional analysis, is given in Section 6.4.

6.2 Dedekind Complete, Ordered, Relational Structures

As we pointed out in the introduction, Archimedeaness captures the commensurability but not the completeness (as far as limits of bounded sequences go) of the real numbers. That is embodied in the idea of Dedekind completeness (or equivalently the existence of least upper bounds within the domain). In this

section, we explore that property as a source of structure and how it interrelates with homogeneity and uniqueness.

THEOREM 6.2 (Luce, 1987, Theorem 4.1) *Suppose $\mathcal{R} = \langle R, \geq, R_j \rangle_{j \in J}$ is a real unit structure. Then the following are true:*

- (i). *\mathcal{R} can be densely imbedded in a Dedekind complete unit structure \mathcal{R}^* , where this imbedding is the identity.*
- (ii). *Each automorphism of \mathcal{R} extends to an automorphism of \mathcal{R}^* .*
- (iii). *If the group T of translations of \mathcal{R} is homogeneous, then the domain of \mathcal{R}^* is Re^+ , and the group T^* of translations of \mathcal{R}^* is homogeneous.*

The next result gives a condition in the Dedekind complete case that leads to the translations being Archimedean.

THEOREM 6.3 (Luce, 1986, Theorem 2.4; Luce, 1987, Theorem 2.1) *Suppose \mathcal{X} is a Dedekind complete, ordered relational structure. If the translations form a group (i.e., are 1-point unique) and are uncrossed, then they form an Archimedean ordered group. A sufficient condition for them to be uncrossed is that \mathcal{X} also be order dense.*

If homogeneity is also satisfied, then we know by Theorem 6.1 that the structure is isomorphic to a homogeneous unit structure.

Our last result on Dedekind complete structures considers what happens when we add the further condition that the structure is unique. We do not know very much about the uniqueness of general real unit structures, but in the homogeneous case, we have strong results. (Note that the following theorem states that the translations form a group, which is surprisingly hard to show.)

THEOREM 6.4 (Alper, 1987, Theorem 3.10) *Suppose \mathcal{X} is a relational structure that is Dedekind complete, order-dense, homogeneous, and unique. Then the translations form a homogeneous, Archimedean ordered group, and \mathcal{X} is isomorphic to a real unit structure that has a subgroup of the nonnegative affine transformations as its automorphism group. Thus, the structure is 1- or 2-point unique.*

This important result is the culmination of work begun by Narens (1981a, 1981b) for the (M, M) scale types and extended by Alper (1985) to the $(M, M + 1)$ case. What it shows is that homogeneous and unique structures that can be mapped onto the reals are of just three scale types—(1, 1) or ratio scale, (2, 2) or interval scale, and the in-between (1, 2) case. An example of the later is the group of discrete interval scales of the form $x \rightarrow k^n x + s$, where $k > 0$ is fixed and n ranges over the integers.

It is clear, especially in view of Theorem 6.1, that we should try to gain a

better understanding of the uniqueness of homogeneous, real unit structures whose domains are not Re^+ .

6.3 Intrinsic ds -Archimedeaness

We may use the previous results, in particular Theorems 4.3, the corollary to Theorem 6.1, and Theorem 6.4, to get the following, quite general characterization of intrinsic ds -Archimedeaness.

THEOREM 6.5 *Suppose $\mathcal{X} = \langle X, \succeq, R_j \rangle_{j \in J}$ is a relational structure that is homogeneous and finitely unique, and suppose $\langle X, \succeq \rangle$ is a continuum. Then \mathcal{X} is intrinsically ds -Archimedean.*

It is important to note that, for the property of ds -Archimedeaness to be nontrivial, it is necessary that an unbounded ds -sequence exist, and, for that to be so, some form of solvability must be satisfied. This means that Theorems 6.5 and 5.1, although apparently parallel, have somewhat different significance, because, in the positive case, both ss - and ds -sequences always exist.

It should also be pointed out that we do not have any generalization of Theorem 6.5 to, for example, structures that are not homogeneous but whose translations continue to form an Archimedean ordered group or to structures not on a continuum. Presumably some of these should continue to be considered to be Archimedean in some sense.

6.4 Distribution of Unit Structures in Conjoint Ones

Suppose $\mathcal{C} = \langle X \times P, \succeq \rangle$, where \succeq is a weak ordering (i.e., transitive and connected) of $X \times P$. \mathcal{C} is said to be a *conjoint structure* if and only if \succeq exhibits monotonicity in each of the two factors (this is often called “independence”) and, therefore, in an obvious way, induces weak orderings \succeq_X on X and \succeq_P on P . A sequence $\{x_i\}$ from X is said to be *standard* if and only if there are p, q in P that are not equivalent under \sim_p such that, for x_i and x_{i+1} in the sequence, $(x_{i+1}, p) \sim (x_i, q)$. \mathcal{C} is said to be *Archimedean* if and only if each bounded standard sequence is finite. There are a number of notions of solvability for conjoint structures, the simplest and strongest being that, given any three of x, y in X and p, q in P , the fourth exists that solves the equivalence $(x, p) \sim (y, q)$. This form is called *unrestricted solvability* (Luce and Tukey, 1964).

There have been two lines of work connecting conjoint structures with other structures. One, which we do not go into here, involves recoding the information embodied in the conjoint structure as an operation on one of its components. In the Archimedean case, these operations are very closely connected to PCSs, and the whole problem of representing conjoint structures numerically is readily reduced to that of PCSs.

The other line of work has been a series of gradually more general results of the following character: Suppose that one of the factors, say X , of an Archimedean conjoint structure has on it a homogeneous structure with a homomorphism ϕ onto a real unit structure. Suppose, further, that this structure is suitably interconnected with the conjoint one according to a concept of distribution. Then the conjoint one is necessarily an “additive conjoint structure” with a multiplicative representation of the form $\phi\psi$. This was first shown to hold when that structure on X is extensive and distribution is defined for operations (Narens, 1976; Narens & Luce, 1976); it was next extended to PCSs (Narens, 1981a; Luce & Cohen, 1983) and then further clarified for general closed concatenation structures (Luce & Narens, 1985) and ultimately put in the form given below for homogeneous real unit structures (Luce, 1987). The interest in the result is mainly embodied in the second result of this section that shows how the structure of physical units can be extended to incorporate these unit structures.

To state the results, we first formulate, in a very general fashion, the notion of a structure on one component being compatible with the conjoint structure. Two n -tuples of X , (x_1, \dots, x_n) and (y_1, \dots, y_n) , are said to be *similar* in \mathcal{C} if and only if there exist p, q in P such that, for each $i = 1, \dots, n$, $(x_i, p) \sim (y_i, q)$. (For example, any n -term subsequence of successive terms of a standard sequence and the subsequence obtained by shifting the indices up by one are similar.) Note that similarity is not transitive. Let S be a relation of order n on X . S is said to *distribute* in \mathcal{C} if and only if, for each similar pair of n -tuples, when one is in S , then so is the other. An ordered relational structure $\mathcal{X} = \langle X, \succeq_X, S_j \rangle_{j \in J}$ is said to distribute in \mathcal{C} if and only if, for each j in J , S_j distributes in \mathcal{C} . (Note it is easy to show that \succeq_X always distributes in \mathcal{C} .)

THEOREM 6.6 [Luce, 1987, Theorem 5.1.(ii) and (iii)] *Suppose \mathcal{X} is a densely ordered relational structure and its set of translations, \mathcal{T} , forms a group. Then \mathcal{T} is a homogeneous, Archimedean ordered group if and only if there exists an Archimedean, solvable, conjoint structure C and a relational structure \mathcal{X}' on X such that \mathcal{X} and \mathcal{X}' are isomorphic and \mathcal{X}' distributes in C . In this case, C satisfies the Thomsen condition⁶ leading to a multiplicative representation.*

Note that the condition of \mathcal{T} being a homogeneous, Archimedean ordered group is identical to that of Theorem 6.1, and, therefore, it says that a homogeneous, real unit structure is distributive in some multiplicative conjoint structure. Thus, the two results bring together three important ideas—real unit structures, distribution in a conjoint structure, and Archimedean ordered translations—that are not obviously linked, and it shows that, in the homogeneous case, they are equivalent ideas.

Perhaps the most important aspect of this is that it makes clear the circum-

⁶If $(x, r) \sim (u, q)$ and $(u, p) \sim (y, r)$, then $(x, p) \sim (y, q)$.

stances under which nonadditive, numerical scales can be introduced into the dimensional structure of physics. To this end, we have the following result.

THEOREM 6.7 (Luce, 1987, Theorem 5.2). *Suppose $\mathcal{C} = \langle X \times P, \succeq \rangle$ is a conjoint structure that is solvable and Archimedean. Suppose, further, that $\mathcal{X} = \langle X, \succeq_X, S_j \rangle_{j \in J}$ is a relational structure whose translations form an Archimedean ordered group.*

- (i). *If \mathcal{X} is distributive in \mathcal{C} , then \mathcal{X} is 1-point homogeneous and \mathcal{C} satisfies the Thomsen condition.*
- (ii). *If, in addition, \mathcal{X} is Dedekind complete and order dense, then, under some mapping ϕ from X onto Re^+ , \mathcal{X} has a homogeneous unit representation and there exists a mapping ψ from P into Re^+ such that $\phi\psi$ is a representation of \mathcal{C} .*
- (iii). *If, further, there is a Dedekind complete relational structure on P that distributes in an analogous way in \mathcal{C} and there exists a homogeneous unit representation ψ , then there exists a real constant ρ such that $\phi\psi^\rho$ is a representation of \mathcal{C} .*

This yields the familiar representation of units of measurement as products of powers of other units that underlies the method of dimensional analysis (see Krantz, Luce, Suppes, & Tversky, 1971, chapter 10; Luce, 1978).

7. NON-ARCHIMEDEAN STRUCTURES

We will now discuss very briefly some of the metamathematical results that apply to the axiomatization of Archimedean structures. The first, and perhaps the most profound, is that Archimedeaness can never be expressed or even implied through first-order sentences. The proof is a very straightforward consequence of the Löwenheim–Skolem Theorem⁷ of mathematical logic and does not depend in any interesting way on the particular concept of Archimedeaness used.

Let $\mathcal{X} = \langle X, \succeq, R_1, \dots, R_n \rangle$ be an ordered relational structure. To make matters interesting, we assume X is infinite. Let $\mathcal{L} = \mathcal{L}(\succeq, R_1, \dots, R_n)$ be a first-order language that describes \mathcal{X} . Then, by the Upward Löwenheim–Skolem Theorem, it follows that there exists a class \mathcal{M} of models of \mathcal{L} of arbitrarily high cardinality that have exactly the same true statements in \mathcal{L} as \mathcal{X} . Because \mathcal{X} is totally ordered and a “total ordering” is expressible in \mathcal{L} , it follows that each model in \mathcal{M} that has cardinality greater than the reals cannot be imbedded in a structure based on the reals, and, therefore, cannot be imbedded in any structure based on a continuum. Thus, in particular, they cannot be imbedded in any intrinsic Archimedean structure. Imbeddability into an intrinsic Archimedean structure is taken as an essential condition of Archimedeaness; therefore, it

⁷Exact statement and proof of the Löwenheim–Skolem Theorem can be found in Narens (1985), Skala (1975), and Robinson (1963).

follows that those elements of \mathcal{M} with domains of high cardinalities cannot be Archimedean. Furthermore, it is easy to show that there exists arbitrarily large $\mathcal{X}' = \langle X', \succ', R_1', \dots, R_n' \rangle$ in \mathcal{M} (i.e., those that are “not cofinal with ω ”) such that, for each strictly increasing, positive operation \circ on X' and each x in X' , there exists y in X' such that, for all positive integers n , $y \succ' nx$. (Note that it is not assumed that \circ is monotonic or invariant under automorphisms of \mathcal{X}' .)

The previous results show that no reasonable concept of Archimedeaness can be captured in a first-order way. Thus, those who hold that empirical concepts can always be formulated in a first-order language must accept Archimedeaness as being necessarily nonempirical.

The assumption of Archimedeaness, however, often has empirical consequences. The class \mathcal{C} of positive, associative, restrictedly solvable concatenation structures is a good example of this. This class is axiomatizable through a first-order language. Some of the structures in \mathcal{C} are Archimedean and represent widely used, important empirical situations; others are non-Archimedean. The following can be shown: (a) for structures in \mathcal{C} , *ss*-Archimedeaness implies commutativity, and (b) there are structures in \mathcal{C} that are not commutative. Note that it follows from (a) and (b) that, for elements of \mathcal{C} , noncommutativity—a first-order and often an empirically verifiable condition—implies non-Archimedeaness. However, it can also be shown (using the Robinson model completeness test) that the subclass \mathcal{D} of elements of \mathcal{C} that are divisible—that is, first-order statements of the form $\forall x \exists y (ny = x)$ are true for all positive integers n —commutative and solvable have the same first-order consequences in the presence of the assumption of Archimedeaness as they would without it. In fact, it can be shown that the theory \mathcal{D} is complete in the sense that a first-order sentence is true about one of its elements if and only if it is true about all of its elements.

The upshot of these results can be summarized as follows. If we take first-order expressibility as a necessary characteristic of empirical, then we have the following: (a) Archimedeaness (in an infinite setting) is never an empirical consequence; (b) non-Archimedeaness is sometimes an empirical consequence; (c) in some empirical situations, the assumption of Archimedeaness adds new empirical consequences; and (d) in some empirical situations, the assumption of Archimedeaness adds no new empirical consequences. Clearly statement (d) characterizes a highly desirable state of affairs. We do not know, however, how generally it applies to situations that one is likely to encounter in science.

8. CONCLUSIONS

The issue addressed in this paper is what Archimedeaness might mean for ordered structures that may not include operations. The solution proposed first involves defining the intrinsically Archimedean structures. These we take to have

continua as domains. Structures that are densely imbeddable in them are considered to be the Archimedean ones, or, in other words, a structure is said to be Archimedean if and only if it has a Dedekind completion that is an intrinsic Archimedean structure. Therefore, the issue is what the latter means.

Intrinsically Archimedean structures we take to be ones to which we can adjoin operations that have two essential features: (a) They have enough structure themselves to be viewed as Archimedean (e.g., they are monotonic and satisfy some strong form of Archimedeaness); and (b) they are invariant under the automorphisms of the given structure. The justification for (b) rests on the observation that all operations definable in terms of the primitives of the structure are invariant under the automorphisms of the structure, so such invariance can be viewed as a generalization of definability.

For homogeneous structures on the continuum, we argue that the class of unit structures are suitable for testing the Archimedeaness of the given structure. In the positive case, these structures can be shown to satisfy all three versions of Archimedeaness that have been proposed for positive structures: in standard sequences, in standard differences, and in difference sequences. In the idempotent case, we do not have equally satisfactory results, partly because our current concepts of Archimedeaness for this case are nontrivial only if a solvability condition is satisfied.

With these unit structures as our criterion, it can then be shown that a homogeneous ordered structure on a continuum is Archimedean if and only if its translations (i.e., automorphisms with no fixed point plus the identity) form an Archimedean ordered group (where the group ordering is naturally induced from the ordering of the structure). This kind of structure has been shown to have nice numerical representations and to be exactly the class of structures that can be incorporated into the structure of dimensions that arose in classical physics.

Although we have succeeded in pinning down a general and sensible concept of Archimedeaness for those structures that can be extended to a homogeneous structure on a continuum, we do not at this time know much at all about the Archimedeaness of structures that cannot be so extended.

ACKNOWLEDGMENTS

This work has been supported by National Science Foundation Grant IST-8602765 to Harvard University.

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