JOURNAL OF MATHEMATICAL PSYCHOLOGY **40**, 78–98 (1996)
ARTICLE NO. 0005

The Ongoing Dialog between Empirical Science and Measurement Theory*

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This review article attempts to highlight from my personal perspective some of the major developments in the representational theory of measurement during the past 50 years. Emphasis is placed on the ongoing interplay between the development of abstract theory and the attempts to apply it to empirically testable phenomena. The article has four major sections. The first concerns classical representational measurement, which was the successful attempt to formulate the major measurement methods of classical physics: extensive and additive conjoint structures, their distributive interlock in dimensional analysis, and intensive (averaging) structures. The second illustrates a nontrivial behavioral example using both extensive and conjoint measurement plus functional equations to arrive at rank- and sign-dependent utility (also called cumulative prospect) representations for decision making under risk. The third section, contemporary representational measurement, somewhat overlaps the classical one but includes new findings and approaches: representations of nonadditive concatenation and conjoint structures; a general theory of scale types; results for general, finitely unique, homogeneous structures; structures that are homogeneous between singular points; generalized distributive triples; and a generalization of dimensional analysis to include any ratio scalable attribute; and the concept of meaningfulness. The final section concerns applications of the latter ideas to psychophysical scaling and merging functions.

1. INTRODUCTION

The dialog between empirical science and measurement theory is, of course, exceedingly complex and detailed, but for the purposes of an overview I think that one can divide it crudely into three overlapping aspects. The first, which I call *classical representation measurement* (Section 2), can be thought of as being predominant in the period from, say, von Helmholtz's 1887 paper, which in English translation was called "Counting and Measuring" (1887/1930), through the publication in 1971 of both Volume I of the *Foundation of Measurement* (Krantz, Luce, Suppes, & Tversky, 1971)[1] and Pfanzagl's *Theory of Measurement* or even to Roberts' (1979) *Measurement Theory* which bridges to the newer material. It is always a question whether or not to treat geometry as part of classical measurement. Although I think it is quite reasonable to do so—witness FM-2—for the purpose of this article I have chosen not to include it.[2] Classical measurement theory is characterized primarily by topics in physical measurement leading to representations in what became in the 19th century well understood numerical systems of the types widely employed in classical physics. The effort in this phase was primarily to understand the empirical source of these numbers. For some general criticisms of the representational measurement approach, see Savage and Ehrlich (1992).

It is overly simple to say that all of the classical phase was completed in 1971 or 1979 because discussions continue about how best to view the generation of numbers from measurement (Michell, 1990; Niederée, 1987, 1992)—indeed, one session of the Kiel conference was devoted to this topic. Another topic, long an issue in the classical literature, which has also received some attention in the past 24 years, is error or uncertainty in measurement. Although the approaches so far taken seem classical in nature, it could well be that something quite distinct from classical methods is needed to increase their usefulness.

The second phase, which I refer to as *contemporary representational measurement* (Section 4), became active in the mid 1970s and continues to the present; however, its antecedents began at least as early as the famous Stevens

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* This article is based on an invited address given at the conference *Foundations of Measurement*: *The Theory of Representability and the Nature of Numbers*, Kiel, Germany, September 21–24, 1995, which was organized by Dieter Heyer, Rainer Mausfeld, and Reinhard Niederée. I appreciate the comments and suggestions of William K. Estes, A. A. J. Marley, Louis Narens, and Thomas Wallsten on earlier drafts. This work was supported in part by National Science Foundation Grants SES-930859 and SBR-9520107 to the University of California, Irvine.

[1] For brevity, this volume will be referred to as FM-1; FM-2 is Suppes, Krantz, Luce, & Tversky (1989), and FM-3 is Luce, Krantz, Suppes, & Tversky (1990).

[2] Mundy (1994) takes to task representational measurement theorists, at least to the extent they see themselves as philosophers of science, for adopting an approach to measurement that is too piecemeal. He emphasized that our formulations are not sufficiently sweeping to include the various representations used in modern physics.

0022-2496/96 $18.00

(1946) article pointing out the limited number of scale types found in physical measurement. This phase has three interrelated and still developing themes: nonadditivity, scale types, and meaningfulness. (As was noted, it is unclear whether the treatment of error will eventually become a part of this movement.) This phase is characterized by the development of new mathematical generalizations and by an improved mathematical understanding of what underlies representational measurement. Its purpose is primarily to inform the sciences—especially the behavioral and social ones—of the full range of measurement possibilities and to provide concrete contributions to those sciences. Although parts of this development seem fairly definitive, others are clearly far from completed and are being actively pursued.

The third aspect consists of attempts to put our knowledge about measurement to use in devising theory in some of the sciences. In my opinion, we have only seen the tip of the iceberg as far as scientific applications are concerned. My coverage of them does not try to be comprehensive. Some references are cited to other applications, and Roberts (1979) included a number of the earlier applications. What I do cover is broken into two sections. The first, Section 3, concerns utility theory, to which I have devoted considerable attention in the past few years, and it draws primarily on the classical results. The second, Section 5, concerns psychophysics and the merging of measurement scales, and it draws primarily on the more recent results.

2. CLASSICAL REPRESENTATIONAL MEASUREMENT

The theme of classical measurement is the construction of additive and vector space representations. In my view, the major topics are extensive measurement as the source of addition; additive conjoint structures and trade-off (or conservation) laws; the distributive interlock between extensive and conjoint structures; dimensional analysis and its relation to fundamental measurement; and finally intensive structures.

2.1. *Extensive Measurement and Numbers*

Although historians have traced developments in theory of measurement back to the Greeks, Helmholtz (1887/1930) is one of the earliest attempts to state explicitly—in terms of an ordered system with an empirical operations—the collection of empirical laws that give rise to an additive numerical representation. Hölder (1901) placed this system on an even better mathematical basis, and others later extracted from his paper the theorem now named for him: Every Archimedean ordered group is isomorphic to a subgroup of the additive reals (or, equally, of the multiplicative positive reals).

Such structures, called extensive, consist of a set A of elements that exhibit the attribute to be measured; an ordering of A, $\succsim$, by that attribute—so if $a, b \in A$, $a \succsim b$ means that a exhibits at least as much of the attribute as does b; and a binary operation $\circ$ on A of combining elements so that $a \circ b$ also exhibits the attribute. Mass and length are prototypical. These primitives are assumed to satisfy the following conditions[3]: $\succsim$ is a weak order, and the relation between $\succsim$ and $\circ$ is monotonic, associative, commutative, positive, restrictedly solvable, and Archimedean.

In a resurgence of interest in the middle part of the century and extending to the present, various improvements and generalizations of extensive measurement structures were developed. Among these developments were improved versions of the classical axiom systems (Niedereée, 1987, 1992; Roberts & Luce, 1968; Suppes, 1951); their application to difference and absolute difference structures (Debreu, 1958; Scott & Suppes, 1958; Suppes & Winet, 1955; Tversky & Krantz, 1970); and modifications of the classical model to bounded (Luce & Marley, 1969) and periodic structures (Luce, 1971b).

There is little doubt that variants of extensive structures form suitable models for such physical attributes as length, mass, time, angle, etc., although not of density, momentum, hardness, etc., and that these empirical systems are probably the major source of the numerical ideas of order and of addition of, at least, the integers and rationals. Although many of us today, e.g., FM-1, 2, 3, may appear to treat numerical systems platonically—as an abstract concept given a priori—I think that most of us recognize, as Michell (1990) has been at pains to point out, that the additive rational numbers really arise as an abstract formalization of the structure exhibited by many physical attributes.

2.2. *Additive and Nonadditive Representations of Extensive Structures*

An extensive structure $\langle A, \succsim, \circ \rangle$ always has an additive representation, i.e., a mapping ϕ into the real numbers such that for all $a, b \in A$,

$$a \succsim b \Leftrightarrow \phi(a) \geqslant \phi(b) \tag{1a}$$

$$\phi(a \circ b) = \phi(a) + \phi(b). \tag{1b}$$

Often this is the representation that is used—length, mass, and charge are examples. But an extensive structure also has

[3] $\succsim$ is a weak order if it is connected and transitive; the relation between $\succsim$ and $\circ$ is monotonic $\Leftrightarrow \forall x, y, z,\ x \succsim y$ iff $x \circ z \succsim y \circ z$ iff $z \circ x \succsim z \circ y$; associative $\Leftrightarrow \forall x, u, z,\ x \circ (y \circ z) \sim (x \circ y) \circ z$; commutative $\Leftrightarrow \forall x, y$, $x \circ y \sim y \circ x$; positive $\Leftrightarrow \forall x, y, x \circ y \succ x$ and $\succ y$: restrictedly solvable $\Leftrightarrow \forall x, y$, if $x \succ y$, then $\exists z$ such that $x \succsim y \circ z$; and Archimedean $\Leftrightarrow \forall x, y, \exists$ integer n such that $nx \succ y$, where $1x = x$ and $nx = (n-1)\, x \circ x$. Commutativity can be deduced from the other properties. Note that all of these properties, save solvability, are necessary if an additive representation on the positive real numbers exists.

a multitude of nonadditive ones—indeed, any strictly increasing function g on the reals generates one. And, sometimes, scientists opt to use one of the nonadditive representations. Perhaps the most vivid example is the "adding" of velocities in the theory of special relativity. In this case the qualitative structure of velocity concatenation is extensive except for having a maximal element, the velocity of light. Omitting that value, the remaining structure is extensive. With velocity given its usual definition in terms of distance s and time t, $v = s/t$, then it turns out that the extensive operation is represented as:

$$u \oplus v = \frac{u+v}{1+uv/c^2}, \tag{2}$$

where c is the velocity of light in that representation. (It is ∞ in the additive representation.) This result was dictated primarily by certain physical principles of invariance that I do not go into here (but see Mundy, 1986, 1994).

A considerably less familiar example of a somewhat analogous situation has arisen recently in utility theory (Section 3.2). I believe that it is important for behavioral scientists to realize that the existence of an additive representation does not mandate its use if some nonadditive one meshes better with other structures in which the extensive attribute plays a role.

2.3. *Additive Conjoint Structures and Conservation Laws*

Helmholtz (1887/1930) remarked, as did many later authors, on how multiplication, as well as addition, appears in physics when different dimensions are involved. In translation (p. 29–30), he said:

> "The multiplication of denominate numbers by pure numbers remains then wholly within the limits of the definitions and propositions which are deduced above for the multiplication of pure numbers among themselves.
>
> It is otherwise with the multiplication of two or more denominate numbers. This has meaning only in definite cases when special physical combinations are possible among the units involved, which (combinations) are subject to the three laws of multiplication...
>
> Physics forms a great number of such products of different units and corresponding thereto also examples of quotients, powers, and roots of the same...
>
> Most of these combinations rest upon the determination of coefficients; many of these magnitudes can supply, however, in addition additive physical combinations such as rates of motion, currents, forces, pressures, densities, etc."

Subsequent authors, including Campbell (1920, 1928) and Bridgman (1922/1931), continued to treat multiplication between attributes as both important but secondary to extensive measurement. Indeed, such multiplicative measures were called *derived* because of their apparent dependence on prior extensive measurements. An excellent example is the density ρ of a homogeneous substance which involves extensive measures of the volume, V, and the corresponding mass, $m(V)$, of the substance. The empirical law is that for any substance the ratio $\rho = m(V)/V$ is independent of V.

In the 1960s a new tack was developed in which ordered Cartesian products came to be studied in their own right as embodying scientifically important trade-offs (or conservation laws) between independent variables affecting an attribute of interest (Debreu, 1960; Luce & Tukey, 1964). The motive was to generalize measurement to the behavioral and social sciences where extensive operations seemed sparse, but once formulated we recognized that this mathematical system also captured, in as fundamental a way as extensive measurement, the qualitative features of the so-called derived measures. For a fairly recent review, see Wakker (1989).

For $\langle A \times P, \succsim \rangle$ to have a numerical additive representation $\phi_A + \phi_P$ that preserves the order $\succsim$ major necessary properties are: $\succsim$ is a weak order that is monotonic in each coordinate, each coordinate matters, Archimedeaness holds in some suitable sense, and crucially that the Thomsen condition holds: $\forall a, b, f \in A, p, q, x \in P$

$$(a, x) \sim (f, q) \quad \text{and} \quad (f, p) \sim (b, x) \Rightarrow (a, p) \sim (b, q). \tag{3}$$

To achieve sufficiency, we added a form of solvability whose details will become apparent shortly. After being exposed to this result, Krantz (1964) quickly saw that the mathematical proof could be reduced to that of extensive measurement, and Holman (1971) provided an alternative and ultimately more useful definition of an extensive operation that encodes on one of the independent variables, say A, all of the information found in the trade-off structure. To be explicit, suppose $\langle A \times P, \succsim \rangle$ is the conjoint structure and $a_0 \in A$, $p_0 \in P$ are fixed elements. For $b \in A$, define $\pi(b)$ to be a solution to the equivalence

$$(a_0, \pi(b)) \sim (b, p_0). \tag{4a}$$

Clearly, one must assume that such solutions exist (but see Section 2.7 for cases where they need not exist). In words, the "interval" from a_0 to b on the A component is "matched" by the "interval" from p_0 to $\pi(b)$ on the P component. In terms of this, for $a, b \in A$, define $a \circ b$ to be a solution, again assumed to exist, to the equivalence:

$$(a \circ b, p_0) \sim (a_0, \pi(b)). \tag{4b}$$

In words, the "interval" from a_0 to $a \circ b$ is the "sum" of the (a_0, a) and (a_0, b) intervals achieved by first mapping the "interval" (a_0, b) of A to the equivalent "interval" $(p_0, \pi(b))$ and then by an inverse transform map that to the equivalent "interval" $(a, a \circ b)$. For structures with a strong form of solvability, the operation is always defined; with more restricted forms of solvability, only a partial operation arises, but the theory of extensive structures had been generalized to cover that case as well (Luce & Marley, 1969).

Additive conjoint structures have come to be moderately important in the behavioral sciences in at least three ways. First, they correspond closely to the scientific strategy of studying trade-offs among independent variables that maintain a dependent attribute constant. Psychologists and economists, in particular, had long exploited this strategy in the form of equal-attribute contours or indifference curves (see Levine, 1971, 1972; Krantz & Tversky, 1971; Michell, 1987, 1990; and any introductory economics text). Second, this development helped initiate a cottage industry of business applications. Two fairly recent summary references are Green and Srinivasan (1990) and Wittink and Cattin (1989). And third, they serve as partial underpinnings for theories such as the utility one discussed in Section 3.

2.4. *Distributive Triples*

Once additive conjoint measurement was axiomatized, we realized that some attributes would necessarily have two quite independent axiomatizations. Consider mass as an example. Of course, the extensive operation of combining masses leads to an additive representation; an equally good axiomatization arises in the conjoint structure whose components are volumes and homogeneous substances. Clearly, physics admits but one measure of mass, not two independent ones, so there has to be an interlock that captures that fact. Formulating that interlock was first addressed by Luce (1965) and Marley (1968) (and reported in FM-1, 1971), but the major improvement in understanding came with Narens' (1976) formulation of distribution[4] in a utility context, which Luce and Narens (1985) later used much more generally. The idea is that if $\succsim$ is an ordering of a Cartesian product $A \times P$ and $\circ_A$ is a binary operation on A, then the structure is *distributive* if

$$(a, p) \sim (c, q) \text{ and } (b, p) \sim (d, q)$$
$$\text{imply } (a \circ_A b, p) \sim (c \circ_A d, q). \tag{5}$$

Suppose that the conjoint structure is additive, that $\langle A, \succsim_A, \circ_A \rangle$ is extensive with an additive representation ϕ_A onto the positive reals, $\mathbb{R}^+$, and that the extensive structure is distributive in the conjoint one. Then we showed that there is an order preserving mapping ϕ_P from P into $\mathbb{R}^+$ such that $\phi_A \phi_P$ represents the conjoint structure. Thus, the extensive and conjoint measures of the A-attribute agree. Such interlocked structures were called *distributive triples*.

[4] In arithmetic the distributive law is $(x+y)\,z = xz + yz$.

In cases where extensive operations also exist on at least one of P and $A \times P$, then the conjoint representation takes the form of either $\phi_A \phi_P^k$ or $\phi_A \phi_{A \times P}^k$. In these cases one can state precise qualitatively laws (of exchange and similitude) that serve both to impose distribution and to characterize k as a rational number.

2.5. *Dimensional Analysis*

Beginning at least as early as 1822 in Fourier's classical work on heat, both applied physicists and engineers have employed the method of dimensional analysis to uncover the mathematical form of physical laws. A key statement of the method was Buckingham (1914), and careful vector space formulations can be found in Palacios (1964), Sedov (1956/1959), and Whitney (1968). Of these, I think Whitney did it best. Although everyone recognized that extensive structures serve as the basis of the vector space of physical quantities and that derived measures are linked as products-of-powers to extensive representations and to other derived ones, no actual detailed formalization of this interlock was provided. Rather, it was simply assumed that a numerical vector space representation was suitable, with little suggestion of how it was to be constructed qualitatively from extensive structures. Once the qualitative theory of distributive triples was worked out, it became feasible to provide an explicit account. A first attempt at accounting for such a representation was Causey (1969); a somewhat different version was presented in Chapter 10 of FM-1; and an improved version was reported in Section 22.7 of FM-3. The crucial linkage is having a sufficient number of distributive triples: every nonratio scaled attribute must appear in at least one distributive triple with an extensive attribute. Moreover, two triples that involve the same variables, such as mass and velocity in laws for both momentum and kinetic energy, must be dimensionally consistent.

Such a formulation in terms of distributive triples appears to cover the classical structure of physical quantities, but it fails to account for such relativistic ones as velocity. It is easy to show from Eq. (2) and $s = vt$ that the distributive interlock, Eq. (5), fails. So far, no one has suggested an effective way to describe the interlock in such cases. To do so is of considerable importance because attributes such as utility, loudness, and brightness all seem to be bounded. Indeed, as we will see in Section 3.2, the boundedness of utility is readily forced by other plausible conditions. I believe that the behavioral sciences will benefit considerably when we understand better the nature of the interlock between conjoint structures, with a multiplicative representation, and operations on one of its components with a representation, in terms of the corresponding conjoint measure, that is bounded and nonadditive.

Returning to dimensional analysis, a crucial source of its power to arrive at the form of laws is the postulate or principle that any physical law must exhibit *dimensional invariance*, which asserts that if the variables $x_1, x_2, ..., x_n$ satisfy an empirical law of the form $F(x_1, x_2, ..., x_n) = 0$ and if ϕ is what physicists call a *similarity transformation* of the variables, then it is also the case that $F[\phi(x_1), \phi(x_2), ..., \phi(x_n)] = 0$. What is surprising about dimensional invariance, and is best appreciated only by studying actual examples of its use, is the degree to which this principle determines the form of physical laws *when*, *but only when*, one knows precisely which variables and dimensional constants are relevant.

In FM-1 we recounted three distinct arguments that have been offered in defense of assuming dimensional invariance, but we were not content with any of them. Later work, discussed below in Section 4.8, seemed to clarify why such a property is a necessary (although certainly not a sufficient) condition for a function among variables to be a physical law.

2.6. *Intensive Bisymmetric Structures and Averaging Representations*

A second major class of physical structures with an operation are those for which an averaging representation holds, such as

$$\phi(a \circ b) = p\phi(a) + (1-p)\,\phi(b), \tag{6}$$

for some fixed $p \in (0, 1)$. The major qualitative features that distinguish such systems from extensive ones are: they are intern, i.e., $\min(a, b) \precsim a \circ b \precsim \max(a, b)$, rather than positive, and associativity does not hold but is replaced by the following *bisymmetry* property

$$(a \circ b) \circ (c \circ d) \sim (a \circ c) \circ (b \circ d). \tag{7}$$

Pfanzagl (1959) and Aczél (1948) found axioms sufficient for the representation of Eq. (6), and FM-1 proved that result by reducing it to a case of an additive conjoint measurement by defining the following order $\succsim'$ on $A \times A$:

$$(a, b) \succsim' (c, d) \qquad \text{if and only if} \quad a \circ b \succsim c \circ d. \tag{8}$$

Such intensive structures not only play a role in physics, but also in psychology and the social sciences where they are fairly common. One example is the expectation-type representations that have arisen in utility theory (see Section 3); another is the comprehensive work of Anderson (1981, 1982, 1991a, b, c) in which he has fit with considerable success more general weighted average representations to ranking data of various attributes (in this connection, see Section 5.2).

In summary, then, we have reduced the study of averaging structures to that of additive conjoint ones (via Eq. (8)); the study of additive conjoint ones to extensive ones (via Eq. (4)); and the latter to an application of Hölder's theorem by means of an embedding argument. In that chain, bisymmetry leads to the Thomsen condition which in turn leads to associativity, which is the major property underlying the additivity of the representation. Indeed, after perusing FM-1, the category theorist Peter Freyd once remarked, apparently in a disparaging way, that classical measurement theory is nothing but applications of Hölder's theorem. As we shall see in Section 4. 2, this continues to be true, but in a more subtle way, even when we study inherently nonadditive structures.

2.7. *Finite and Countable Structures*

Although I have not personally contributed to the topic, a number of papers have focused on the axiomatization of structures defined over finite or countable sets. Basically, these systems are of three distinct types. The first consists of finite or countable structures that can be mapped onto an interval of integers; such structures form, in essence, a single standard sequence $\{nx_0\}$ obtained by combining the least element x_0 with itself n times. The second type of structure is also countable, but it is dense and it maps into the rational numbers. Axiomatizations of these two types both involve classical ideas and could have been developed early on, although they were not. The third type of structure is finite and representations are based on solving systems of linear inequalities. The major results here are two. The first is the well known theorem of Scott (1964), which was proved by using the theorem of the alternative, that provided a family of necessary and sufficient conditions for an additive representation. A somewhat different formulation of the result is given in FM-1. The second major result is the proof by Scott and Suppes (1958) establishing that no finite system of inequalities can suffice for all finite difference structures; the number of inequalities to be checked must grow with the size of the domain. This result was extended to conjoint structures by Titiev (1972). These results are summarized in Chap. 21 of FM-3.

2.8. *Error*

Bridgman (1927/1938, p. 36) was well aware of the problem of measurement error:

> "In most emperical sciences, the penumbra [of uncertainty] is at first prominent, and becomes less important and thinner as the accuracy of physical measurement is increased. In mechanics, for example, the penumbra is at first like a thick obscuring veil at the stage where we measure forces only by our muscular sensations, and gradually is attenuated, as the precision of measurements increases....".

In a similar vein, Poincaré (1929, p. 46) wrote:

> "It happens that we are capable of distinguishing two impressions one from the other, while each is indistinguishable from a third.... Such a statement, translated into symbols, may be written:
>
> $$A = B, \qquad B = C, \qquad A < C.$$
>
> This would be the formula of the physical continuum, as crude experience gives it to us, whence arises an intolerable contradiction that has been obviated by the introduction of the mathematical continuum....
>
> The physical continuum is, so to speak, a nebula not resolved; the most perfect instruments could not attain to its resolution...".

This sort of ordinal fact was studied in the behavioral sciences long before these remarks were written, beginning with Fechner (1860/1966) and leading to the modern theories of constant and random utility models—which, in fact, are not restricted to utility ideas (Falmagne, 1985; Luce, 1959a; Luce & Edwards, 1958; Marley, 1990; Marschak, 1960; Narens, 1994; FM-2, Chap. 17). Failures of discrimination were also later studied in a purely algebraic framework, first as relations called *semiorders* by Luce (1956) and generalized to *interval orders* by Fishburn (1970). From this beginning a substantial literature has ensued [for summaries see Fishburn (1985) and FM-2, Chap. 16]. Some interest now exists in the lattice structure of all possible semiorders on a fixed finite set, and Doignon and Falmagne (in press) have studied stochastic learning processes that might underlie the development of a weak order from the null order by a series of single step changes in semiorders. This approach provides a possible model for the development of preferences.

But order is not enough. There are operations. Continuing the quotation from Bridgman (1938, p. 36):

> "We may now go still further. Operations themselves are, of course, derived from experience, and would be expected to have a nebulous edge of uncertainty. We have to ask such questions as whether *operations* of arithmetic are clean-cut things. Is the operation of multiplying 2 objects by 2 objects a definite operation, with no enveloping haze? All our physical experience convinces us that if there is penumbra about the concept of operations of this sort it is so tenuous as to be negligible, at least for the present; but the question affords an interesting topic for speculation. We also have to ask whether mental operations may similarly be enveloped in haze."

The attempts so far to incorporate operations into either random utility models or semiorders or interval orders have come to very little. Basically, the only tractable case has been one in which a form of Weber's law holds (Falmagne, 1980; Falmagne & Iverson, 1979; Luce, 1973; Narens, 1980). In sum, we do not at present have a very usable theory of error when we are interested in ordered structures that have additional relations or operations.

An interesting recent development is Heyer and Niederée (1989, 1992) who assume that each respondent satisfies an algebraic model of a certain type, but that the population exhibits a distribution over these models. To my knowledge, this approach has yet to be applied to data. And it assumes that each individual's data are generated from an algebraic system and that all of the probability resides in the population, not the individual. That hypothesis seems very suspect to me.

From my perspective, statistics and measurement theory are really two facets of a single problem that has never been fully formulated, namely, structure and variability. Typically, the statistical approach assumes a form for the representation of the structure and attention is focused primarily on formulating in terms of that representation the properties of variability. In contrast, the measurement theorist formulates the structure qualitatively, and derives the form of the representation, but is unable to confront in any deep way the qualitative representation of variability. Thus, as Luce and Narens (1994) noted, the problem is probably the very deep one of finding a suitable qualitative formulation of randomness. Recall that statistical treatments of randomness are all at the numerical (or representational) level—as random variables. To my knowledge, no satisfactory underlying qualitative theory exists,[5] and so it is unclear how one should incorporate the idea of randomness into the kind of qualitative-representational theory that characterizes axiomatic measurement.

3. APPLYING CLASSICAL MEASUREMENT TO INDIVIDUAL DECISION MAKING

Although classical measurement structures are often applied to data analysis—several examples were mentioned earlier—the number of applications to substantive psychological theory is more limited. The two areas that have received the most theoretical input from contemporary measurement theory are psychophysics and judgment/decision making. I will illustrate aspects of this work, but I make no attempt to provide a thorough survey. In this section I focus on some problems in judgment and decision making to which I have contributed during the past six years and that draw, primarily, on classical measurement ideas. In Section 5 I will illustrate some examples that draw primarily on the more recent ideas summarized in Section 4. Although what I report here only uses classical ideas, it has been in fact affected by the our deeper understanding of measurement from the recent work, e.g., the work on structures with singular points (Section 4.5).

3.1. *Gambles and Lotteries*

Starting with von Neumann and Morgenstern (1947), an elaborate development has taken place of, first, theories of expected, then subjective expected utility (Savage, 1954), and more recently varieties of weighted utilities (Fishburn,

[5] One attempt, which I do not find satisfactory, is that of Suppes and Zanotti (1992) (see also Section 16.8 of FM-2).

1988; Quiggin, 1993). The basic primitive has always been a preference order over *gambles* (or uncertain alternatives), which are defined to be functions from finite partitions of events into a set of consequences (totally distinct from the events). If the decision maker is provided with probabilities rather than uncertain events and if the consequences are money, then the gambles are called *lotteries*. Many experiments use lotteries rather than general gambles, but the theory is just about as easy to state for gambles as for lotteries. Under the normative assumptions made, the structure of the gambles coupled with the ordering turned out to be sufficient to yield interval scale representations of utility, where the utility of a gamble is some form of expectation of the utilities of its consequences.

In part as a result of repeated experimental challenges (some are summarized by Schoemaker, 1982, 1990), mostly mounted by psychologists, the models have grown increasing complex and interesting. Summaries of and references to this literature are Luce and von Winterfeldt (1994), Quiggin (1993), and Wakker (1989). In particular, it was increasingly acknowledged that something is wrong with a theory whose structure had the uniqueness of an interval scale—arbitrary zero and unit—because, as everyone is aware, the separation of consequences into gains and losses is significant and entails a special point, called the status quo, that serves as a natural zero and so is, in some sense, singular (see Section 4.5). Markowitz (1952) was possibly the first to construct a mathematical theory in which the status quo played an explicit role, but most economists attempted to skirt around this difficulty by saying that utility is always calculated over total wealth. Edwards (1962) was the first psychologist to emphasize the importance of the status quo. Of course, everyday evidence suggests that although total wealth certainly affects judgments, transactions are nonetheless usually cast as gains and losses relative to the status quo or, sometimes, relative to an aspiration level (Lopes, 1984, 1987), and gains are treated quite differently from losses. It was also apparent to many, although again not to some theoretical economists, that the concept of risk centers not so much on the shape of the utility function as on the different weights assigned to the events giving rise to losses versus those ending up with gains. For someone who is risk averse, a small probability of a large loss looms far larger than the same small probability of a gain. Lopes (1984, 1987) has pressed this view most strongly, but it is implicit in a number of recent developments. An axiomatic theory of risk perception in which the focus was on the differential impact of gains and losses was worked out by Luce and Weber (1986) and tested there and in several papers by Weber and her colleagues (Weber, 1984, 1988; Weber & Bottom, 1989, 1990).

Although Markowitz (1952) may have been the first to take the distinction between gains and losses into serious account, the paper that really put it on the intellectual map was Kahneman's and Tversky's (1979) prospect theory. This theory involved a somewhat ad hoc representation that applied only to special lotteries and that required some "editing" of lotteries before applying the representation. Toward the end of the 1980s both Kahneman and Tversky and, independently, I considered the issue of generalizing the theory to general gambles. Tversky and I both announced substantially the same representation at the 1989 Santa Cruz conference organized by Ward Edwards (see Edwards, 1992). My work appeared in Luce (1991) and Luce and Fishhurn (1991) and theirs in Tversky and Kahneman (1992) and Wakker and Tversky (1993). A summary of the general ideas is provided by Luce and von Winterfeldt (1994).

In these theories, the basic utility representation, U, that preserves the preference order, $\succsim$, over consequences and binary gambles[6] $(x, E; y)$, where $x, y \succsim e =$ status quo, is the rank-dependent one:

$$U(x, E; y) = \begin{cases} U(x)\, W^+(E) + U(y)[1 + W^+(E)], \\ \quad \text{if} \quad x \succsim y \\ U(x)[1 - W^+(\neg E)] + U(y)\, W^+(\neg E), \\ \quad \text{if} \quad x \prec y. \end{cases} \tag{9}$$

Here U denotes the utility function and W^+ the weighting function used with gains. There is a similar expression for losses using a weighting function W^-. As Luce and Narens (1985) (see Section 4.4) showed, there is an important sense in which Eq. (9) is, up to isomorphism, the most general representation for binary gambles that can hold when U is an interval scale.[7] For the mixed case of $x^+ \succsim e \succsim y^-$, the following sign-dependent form arises:

$$U(x^+, E; y^-) = U(x^+)\, W^+(E) + U(y^-)\, W^-(\neg E). \tag{10}$$

Note that in general $W^+(E) + W^-(\neg E) \neq 1$, which is easily shown to force $U(e) = 0$ and thereby makes utility a ratio rather than an interval scale. This example partially motivated the development of the theory of generalized concatenation structures with singular points that is discussed in Section 4.5.

One task of theory is to understand what properties of preference give rise to Eqs. (9) and (10). Wakker and Tversky (1993) provide one axiom system based just on

[6] The symbol $(x, E; y)$, which really abbreviates the notation $(x, E; y, D)$, is interpreted to mean that a chance device whose universal event is $E \cup D$ determines whether the outcome is x or y, with x arising if the event E occurs and y otherwise. The symbols x and y are generic outcomes, not sums of money or other numerically scaled entities. In the text I will write $\neg E$ for D, but one should interpret this only to mean the complement of E relative to the universal event of a particular chance device.

[7] That is, U is unique up to positive affine transformations $x \to rx + s$.

gambles and preference orders. To me it is not very satisfactory because the axioms are formulated in terms of properties that hold only within a region of gambles having the same sign and rank order of consequences, which means that one builds into the axioms one of the main, but odd, features of the representation. In my opinion, this feature should be explained, not assumed. I have approached the task differently by adding a natural primitive that results both in a plausible formulation of editing and arrives at rank and sign dependence without assuming it in any obvious way.

3.2. *Joint Receipt*

The new primitive[8] is the concept of *joint receipt* (JR) of two things. Symbolically, if g and h are two gambles or pure consequences, $g \oplus h$ denotes receiving both of them. Examples abound: purchases, gifts, both checks and bills in the mail, etc.

Three underlying assumptions that we make are: $\oplus$ is commutative, i.e., for all g and h, $g \oplus h \sim h \oplus g$; $\oplus$ is monotonic, i.e., for all f, g, h, $g \gtrsim h$ if and only if $f \oplus g \gtrsim f \oplus h$; and the status quo e leaves g unaffected, i.e., $e \oplus g \sim g \oplus e \sim g$.

A basic behavioral axiom involving joint receipt of gains is *segregation*, which was studied theoretically by Luce (1991) and Luce and Fishburn (1991): For pure consequences x, $y \succ e$,

$$(x, E; e) \oplus y \sim (x \oplus y, E; e \oplus y) \sim (x \oplus y, E; y). \quad (11)$$

This property, in fact, formalizes one kind of "editing" used in Kahneman and Tversky (1979).

In the presence of some structural assumptions concerning the richness of the consequence and event spaces and that $U(x \oplus x) \leqslant 2U(x)$, Luce and Fishburn (1991, 1995) showed that Eqs. (9) and (11) imply for $x \gtrsim e$, $y \gtrsim e$,

$$U(x \oplus y) = U(x) + U(y) - U(x)\ U(y)/C, \quad (12)$$

where $C > 0$. Note that from the monotonicity of $\oplus$, U is a bounded representation in which C is the least upper bound. In fact, the strictly increasing transformation $V = -\ln[1 - U/C]$ leads to an additive representation,

$$V(x \oplus y) = V(x) + V(y), \quad (13)$$

and so $\oplus$ over gains forms an extensive structure. This, of course, is a testable model.

As Tversky and Kahneman (1992) pointed out, it is plausible to suppose that, for money consequences, $x \oplus y = x + y$. Thaler (1985) argued otherwise using an indirect method of asking students how they thought hypothetical people would react to certain more-or-less real world scenarios. He found additivity sustained for losses but it failed for gains and some mixed cases of gains and losses. Cho and Luce (1995) have established directly that, for money, $\oplus = +$ holds for both gains and losses (we did not study the mixed case). This means that for gains and losses separately, $\oplus$ is trivially an extensive structure, thereby justifying Eqs. (12) and (13), and it is easy to see that $V(x) = kx$ and so U must have the following negative exponential form for $x \succ e$:

$$U(x) = \begin{cases} C(1 - e^{-\alpha x}), & C < \infty, \alpha > 0 \\ \alpha x, & C = \infty. \end{cases} \quad (14)$$

A similar development, with different constants, holds for losses. As we shall see the mixed case is different.

Assuming that segregation (Eq. 11) holds, which seems to be reasonably well supported empirically (Cho & Luce, 1995; Cho, Luce, & von Winterfeldt, 1994), then we can have either the additive representation V of joint receipt (Eq. 13) or the usual weighted average representation of a gambles (Eq. 9), but not both. This is somewhat analogous to the case of relativistic velocity where one cannot have both the additive representation of the concatenation operation and the multiplicative one of the conjoint distance/time structure. A notable difference is that there does not seem to be any consequence, comparable to the velocity of light, corresponding to the least upper bound C. Put another way, one cannot achieve the maximum utility.

A utility function U on binary gambles of gains is said to be *separable* if for some weighting function W^+,

$$U(x, E; e) = U(x)\ W^+(E). \quad (15)$$

Obviously, U is separable if it is rank dependent (Eq. 9). Note that separability simply means that the conjoint structure with factors the set $\mathscr{C}^+$ of gains and the set $\mathscr{E}$ of events satisfies the Thomsen condition, Eq. (3), which property is easily seen to amount to the following special case of what is called *event commutativity*:

$$((x, E; e), D; e) \sim ((x, D; e), E; e). \quad (16)$$

[8] I say "new" because it was new in *this* theoretical literature. In fact, Slovic and Lichtenstein (1968) used joint receipts—they called them duplex gambles—in their experiments and Thaler (1985; Thaler & Johnson, 1990; Linville & Fisher, 1991) studied empirically and, to a degree, theoretically the joint receipt of sums of money. He suggested that if $\oplus$ denotes joint receipt and U denotes utility, then for money $U(x \oplus y) = \max(U(x+y), U(x) + U(y)\}$. He discussed the nature of the boundary between the two terms on the right and found data that he felt supported this rule, which he called the *hedonic rule*. Fishburn and Luce (1995) showed that his analysis about the boundary was incorrect and we corrected it as well as axiomatized the representation. As we shall see, I am now convinced that in the context of the usual laboratory experiments this rule is invalid and that, in fact, for $xy > 0$, $x \oplus y = x + y$.

Brothers (1990) and Chung, von Winterfeldt, and Luce (1994) report data sustaining event commutativity, although some earlier data did not (Ronen, 1973).

Luce and Fishburn (1991, 1995) showed, in a context of adequately rich consequences and events, that segregation (Eq. 11), the utility form given in Eq. (12), and separability (Eq. 15) are equivalent to the rank-dependent form of Eq. (9). In some ways, this result provides a nice account of rank dependence except for one thing: How do we know that Eqs. (12) and (15) both hold at once? It could be, for example, that the V of Eq. (13) is separable, in which case U is not. The following empirical condition follows from Eqs. (12) and (15): for each $x \gtrsim e$ and each $E \in \mathscr{E}$ there is an event $D = D(x, E) \in \mathscr{E}$ such that for all $y \gtrsim e$

$$(x \oplus y, E; e) \sim (x, E; e) \oplus (y, D; e) \tag{17}$$

(Luce, submitted). Aczél, Luce, and Maksa (submitted) show that if Eqs. (12), (16), and (17) hold, then there is a U satisfying both Eqs. (12) and (15). No experiments have yet checked the adequacy of Eq. (17).

For mixed gains and losses, we have another empirical property involving $\oplus$ which was first examined by Slovic and Lichtenstein (1968) (see also Payne & Braunstein, 1971) who called it *duplex decomposition*. It says that people treat a gamble of gains and losses as the joint receipt of the gains aspect and, separately and independently evaluated, the losses aspect. They found it was satisfied by their subjects. Formally, for $x^+ \gtrsim e \gtrsim y^-$,

$$(x^+, E; y^-) \sim (x^+, E'; e) \oplus (e, E''; y^-), \tag{18}$$

where E' and E'' mean that E occurs in two independent realizations of the chance event $E \cup \neg E$. In a very real sense, duplex decomposition is the only clearly nonrational assumption in the theory because the "bottom line" of the two sides of Eq. (18) differ: only x^+ or y^- can occur on the left, whereas in addition $x^+ \oplus y^-$ and e can both occur on the right.

Luce (submitted) shows that sign dependence (Eq. 10) and duplex decomposition (Eq. 18) imply a simple additive conjoint form for $\oplus$:

$$U(x^+ \oplus y^-) = U(x^+) + U(y^-). \tag{19}$$

This testable substructure has not yet been studied empirically. I also show that, together, Eq. (19), separability (Eq. 15), and duplex decomposition (Eq. 18) imply the sign-dependent representation of Eq. (10). Again, we seek an observable property that is equivalent to the U of Eq. (19) being separable. It is this: For all $x^+ \gtrsim e \gtrsim y^-$ and $x^+ \oplus y^- \gtrsim e$ and for each $E \in \mathscr{E}$, there is an event $D = D(E) \in \mathscr{E}$ such that

$$(x^+ \oplus y^-, E; 0) \sim (x^+, E; 0) \oplus (y^-, D; 0), \tag{20a}$$

and, similarly, when $x^+ \oplus y^- \lesssim e$ there is event $D' = D'(E)$ such that

$$(x^+ \oplus y^-, E; 0) \sim (x^+, D'; 0) \oplus (y^-, E; 0). \tag{20b}$$

Again, Eq. (20) is too new to have been examined empirically. Aczél *et al.* (submitted) have proved, on the assumption that U satisfies Eq. (19), that Eq. (20) is equivalent to U being separable (Eq. 15).

Assuming that empirical support found for Eqs. (17) and (20) and for the additive conjoint nature of $\langle \mathscr{C}^+ \times \mathscr{C}^-, \gtrsim \rangle$, then since event commutativity, segregation, and duplex decomposition have been sustained, we have an account for the rank- and sign-dependent version of binary utility (Eqs. 9 and 10). The extension of this theory to non-binary gambles is worked out in Luce and Fishburn (1991, 1995) and in a better fashion by Liu (1995).

3.3. *The Role of Functional Equations*

A major mathematical tool used in these theoretical developments is the theory of functional equations (Aczél, 1966, 1987). It is instructive to see how they come about. As an example, consider the last problem where we want to show that if U is a measure satisfying Eq. (19), then Eq. (20) forces U to be separable. Let V be some order preserving measure that is separable (Eq. 15), which amounts to assuming event commutativity (Eq. 16). Since both measures preserve the preference order, they must be related by a strictly increasing function F, i.e., $U = F(V)$. Applying V to Eq. (20a) we have for the left side

$$\begin{aligned} &V(x^+ \oplus y^-, E; e) \\ &\quad = V(x^+ \oplus y^-)\, W^+(E) \\ &\qquad \text{(Eq. 18)} \\ &\quad = F^{-1}[U(x^+ \oplus y^-)]\, W^+(E) \\ &\qquad \text{Definition of } F) \\ &\quad = F^{-1}[U(x^+) + U(y^-)]\, W^+(E) \\ &\qquad \text{(Eq. 19)} \\ &\quad = F^{-1}[F[V(x^+)] + F[V(v^-)]]\, W^+(E) \\ &\qquad \text{(Definition of } F). \end{aligned}$$

Applying V to the right side of Eq. (20a), by a similar sort of argument

$$\begin{aligned} &V[(x^+, E; e) \oplus (y^-, D; e)] \\ &\quad = F^{-1}[F[V(x^+)\, W^+(E)] + F[V(y^-)\, W^-(D)]]. \end{aligned}$$

Note that in Eq. (20a), if we set $x^+ = e$, apply V, and use its separability we see that $W^+(E) = W^-(D)$. Equating the two sides, and setting $X = V(x^+)$, $-Y = V(y^-)$,

$Z = W^{+}(E) = W^{-}(D)$, then F must satisfy the following functional equation:

$$F^{-1}[F(X) + F(-Y)]\, Z = F^{-1}[(XZ) + F(-YZ)].$$

The problem is to show that a power function is the only solution to this functional equation in which case U is separable (Aczél *et al*., submitted).

This example is quite typical of how functional equations arise in measurement theory applications. Measurement theory yields numerical representations, and any additional behavioral constraint is recast as a functional equation holding among the relevant measures. Then the methods of that literature, which often are not obvious, are used to solve the equation, thereby determining the implication of the behavioral constraint on the measures themselves. Indeed, functional equations seem to play a role in psychology somewhat analogous to differential equations in physics. For additional examples, see Aczél (1987).

3.4. *Certainty Equivalents*

In empirically examining some of the behavioral assumptions above, such as event commutativity, segregation, and duplex decomposition, we (Cho, Luce, & von Winterfeldt, 1994; Chung, von Winterfeldt, & Luce, 1994; von Winterfeldt, Chung, Luce, & Cho, submitted) followed this empirical strategy: For each lottery that would appear in the test of a property we attempted to get a good estimate of its certainty equivalent (CE), which is defined to be the sum of money, $\mathrm{CE}(g)$ or $\mathrm{CE}(g \oplus h)$, such that choice indifference holds with g and $g \oplus h$, respectively, i.e.

$$g \sim \mathrm{CE}(g) \qquad \text{and} \qquad g \oplus h \sim \mathrm{CE}(g \oplus h). \tag{21}$$

Approximate indifference between gambles and joint receipt of them was thereby reduced to verifying approximate equality of CEs. We became convinced that simply asking a subject to report CEs is not the same as determining them by a true choice procedure, so we came to use a modified up–down procedure called PEST.[9] Although PEST is tedious and time consuming, we know of no other reliable way to estimate a choice CE, which is the only thing that is appropriate if we are to replace gambles and joint receipts by their CEs when testing choice theories.

This approach led me both to develop a CE version of the rank- and sign-dependent utility theory and to a careful study of the possible interplay of CEs and JRs of gambles (Luce, 1992b, 1995). I end this section with a description of some of these theoretical results and the resulting empirical findings of Cho and Luce (1995).

Five major questions were raised:

Q1. *Is* JR *monotonic relative to the preference ordering among gambles*?

Under plausible assumptions, including that CE is order preserving over both lotteries and their joint receipt, monotonicity was shown to be equivalent to the following property:

$$\mathrm{CE}(g \oplus h) = \mathrm{CE}[\mathrm{CE}(g) \oplus \mathrm{CE}(h)], \tag{22}$$

which is substitutability in the sense that a gamble in a joint receipt can be replaced by its certainty equivalent.

Q2. *For money, does* $\oplus = +$?

That is, for money amounts x and y, does $x \oplus y = x + y$? Certainly this seems plausible, but as was mentioned earlier, Thaler (1985) had reported data to the contrary, at least for gains and mixed gains and losses.

Q3. *Does some version of segregation hold*?

Note that, depending upon the answers to Q1 and Q2, potentially there are four different quantities at issue:

(a) $(x \oplus y, p;\ y)$

(b) $(x, p;\ 0) \oplus y$

(c) $(x + y, p;\ y)$.

(d) $\mathrm{CE}(x, p;\ y) + y$.

Indifference between (a) and (b) is the basic idea of segregation. We call indifference of (c) and (d), *additive segregation*. The pairings (b) $\sim$ (c) and (b) $\sim$ (d) were studied by Cho, *et al*. (1994), and the former but not the latter appeared to be sustained. Note that if $\oplus$ is monotonic and if $\oplus = +$, then (a) $\sim$ (b) implies that all four should be equal.

One possible meaning for the joint receipt of independent lotteries is their convolution, $*$, where lotteries are treated as independent random variables. Indeed, $\oplus = +$ for money and additive segregation, which have both received empirical support, are special cases of convolution. A first question, then, about convolution is:

Q4. *Does joint receipt equal convolution, i.e.*, $\oplus = *$?

If the answers to Q1 and Q4 were both Yes, then we would know that $*$ is monotonic, but if either is No, then we do not know whether or not $*$ is monotonic. So we asked:

Q5. *Is convolution* $*$ *monotonic relative to* $\succsim$?

[9] We did not use the much faster procedure of judged certainty equivalents because, as Bostic, Herrnstein, and Luce (1990) showed, it does not provide a good estimate of choice-based certainty equivalents. Moreover, as Birnbaum (1992) and Mellers, Weiss, and Birnbaum (1992) showed, the judged certainty equivalents violate a basic monotonicity property when one of the consequences is the status quo e. von Winterfeldt *et al.* (submitted) argued that monotonicity holds using PEST, although the data are sufficiently noisy that the case is not airtight.

I proved, under fairly weak assumptions, that monotonicity of $*$ is equivalent to

$$\mathrm{CE}(g * h) = \mathrm{CE}(g) + \mathrm{CE}(h). \tag{23}$$

The following informal argument was offered to suggest that the answers to Q4 and Q5 might well not both be Yes for those people, whom I called "gamblers," who regularly accept lotteries with (small) negative expected values. If such a person is monotonic in convolution and $\oplus = *$, then he or she should also find repeated convolutions of such lotteries acceptable, which seems unlikely to be widespread because the negative expected value grows in proportion to the number of gambles convolved whereas the standard deviation grows only as its square root.

Cho and Luce (1995) ran a suitable empirical study to answer these questions. On the untested assumption that CE, as estimated by PEST, is order preserving over joint receipt,[10] we provided evidence in support of the following answers:

A1. $\oplus$ *is not monotonic over gambles in the sense that Eq.* (22) *was not sustained.*

A2. *Over money*, $\oplus = +$ *for gains and losses.*

As noted above, it seemed important to attempt to partition the subjects into gamblers and nongamblers. To that end, ten gambles with small expected values were presented and we classed as gamblers those subjects who gave positive CEs to 7 or more of them; the balance we called nongamblers.[11] For A1 and A2 there was no distinction for the two groups.

A3. *For nongamblers all versions of segregation were indifferent. For the gamblers* (a) $\sim$ (b) *and* (c) $\sim$ (d), *but the other indifferences did not hold.*

A4. *For gamblers convolution* $* = \oplus$; *for nongamblers* $* \neq \oplus$.

A5. *For gamblers* $*$ *was not monotonic*; *for non gamblers it was.*

The most disturbing finding is that $\oplus$ over gambles is nonmonotonic. If this discovery is sustained in further work, it creates some pretty serious theoretical headaches because we know very little about representations of nonmonotonic operations.

A most intriguing result was the substantial differences between the subjects empirically classed as gamblers and nongamblers. These differences strongly suggest that we must be very careful about analyzing group data, which are often reported, when testing hypotheses in this domain. One approach to take in studying individuals is that of a psychophysicist who would estimate choice probabilities, construct psychometric functions, and compare them. However, with gambles as stimuli three concerns arise that are more severe than in psychophysics: (1) a major sense of fatigue and boredom resulting from pondering over very many choices; (2) the possibility that subjects may remember specific stimulus pairs and their previous responses to them, and (3) the possibility that to arrive at quick responses over many session subjects, either on their own or with the help of others, will devise simple algorithms that they do not normally use. Despite these fears, Cho, G. Fisher, and I are currently working on collecting enough individual data to study monotonicity of JR and the order preserving aspect of CEs over JR.

If the above results are sustained. then the theoretical development of Luce and Fishburn (1991, 1995) will have to be reexamined. But before doing that, I feel I need to know better than I now do the nature of the nonmonotonicity of joint receipt. Although one can never be confident about future progress, I hope we will have a more complete understanding of the nonmonotonicity issue in the near future.

4. CONTEMPORARY REPRESENTATIONAL MEASUREMENT

I turn now to several nonclassical areas of representational measurement which have been successful in broadening the scope and understanding of representational measurement. Two sources for some of the material covered in this section are FM-3 and Narens (1985); however, the area is active and so both volumes are to a degree out of date.

4.1. *Nonadditive Structures with Positive Operations*

One of the first papers of this new era was Narens and Luce (1976) in which we showed that one could drop the associativity axiom of extensive structures and, changing little else, continue to arrive at a numerical representation. The interest for the behavioral and social sciences in pursuing this direction arises, in part, because, unlike physics, we may be forced to deal with nonadditive operations. We would like to understand how they can be represented numerically.

Although our proof was indirect, via the Cantor–Birkhoff theorem for ordinal structures,[12] a direct proof was subsequently found (FM-3). It is basically similar to that for

[10] Bostic *et al.* (1990) and von Winterfeldt *et al.* (submitted) provide evidence that CE is order preserving for gambles, but the issue of whether CE is order preserving for joint receipt has not been studied.

[11] The data for a break at 8 and at 6 were also reported, and the results are substantially the same. With the criterion at 7, we had 16 gamblers and 24 non-gamblers. A variety of questions aimed at eliciting self reports about gambling behavior were not nearly as useful as this simple behavioral classification.

[12] This theorem states that a weakly ordered system has a numerical, order-preserving representation, Eq. (1a), if and only if it has a countable, order-dense subset.

extensive measurement, but considerably more fussy. Despite certain similarities, the nonadditive result differed in two major respects from the earlier additive ones. First, the representing, nonadditive, numerical mathematics was certainly not provided a priori; rather, it was constructed structure-by-structure, and it was rather arbitrary exactly how that was done. Second, the uniqueness theorem was far less satisfactory than those we have for extensive, conjoint, and intensive measurement. We showed that once a number was assigned to one element of the structure, then the entire representation was completely determined. But this result said nothing whatsoever about how two different representations of a single structure into the same operation are related. We were a bit perplexed.

The situation became clearer during a seminar I taught at Harvard in, if I recall correctly, the spring of 1977. Michael A. Cohen, then a new graduate student in the psychology department, submitted a term paper in which he claimed to prove that the automorphism[13] group of any nonadditive concatenation structure of the type that Narens and I had studied is Archimedean ordered and so, by Hölder, is isomorphic to a subgroup of the multiplicative positive reals. Despite my initial doubts—after all, these nonadditive structures are rather irregular and at the time it seemed unlikely to me that their automorphism groups could all be so regular[14]—I became convinced he was correct. I communicated the result to Narens, who was working on closely related matters, and soon the two of them were collaborating on what became an important paper, Cohen and Narens (1979).

A second major idea of that joint paper, due to Narens, was *homogeneity*. Intuitively, homogeneity means that elements cannot be distinguished structurally one from another. It is a major feature of those extensive structures that have a representation *onto* the reals. The nonadditive representation theorem covered numerous inhomogeneous structures—ones for which elements could be structurally quite different in the sense that for some pairs of elements there was no automorphism mapping one into the other. For example, the entire real numbers under addition fail homogeneity because 0 has the distinctive property that $x+0=0+x=x$, which is not true of any other number. Cohen and Narens showed that the homogeneous positive concatenation structures have a simple numerical representation onto $\langle \mathbb{R}^+, \geqslant, \oplus \rangle$, where $\mathbb{R}^+$ denotes the positive real numbers, $\geqslant$ is ordinary numerical inequality, and $\oplus$ is a binary numerical operation of the form

$$x \oplus y = yf(x/y), \tag{24}$$

for some function f such that $f(z)$ is strictly increasing in z, $f(z)/z$ is strictly decreasing in z, and $f(z) > \max(1, z)$. Moreover, in this representation the automorphisms simply become multiplication by positive constants. Stated another way, the homogeneous structures have a ratio scale representation. Indeed, defining the n-copy operator inductively by $x(n) = x(n-1) \oplus x$, $x(1) = x$, it is fairly easy to show that $x(n)$ is an automorphism, specifically $x(n) = xf^{(n-1)}(1)$. This is a rare case of a structure for which an automorphism can be stated explicitly in terms of the structure itself. We know of nothing comparable for intensive structures although we know how to construct many other automorphisms once one is in hand (Luce & Narens, 1985).

4.2. *Scale Types*

These discoveries opened up a wholly new approach to measurement theory that among other things illuminated the 1930's discussions between psychologists, led by S. S. Stevens, and physicists and philosophers of physics, led by N. R. Campbell, concerning the sources of representational measurement. The latter group contended that extensive empirical operations with additive representations were the sole source of fundamental measurement, and everything else was derived. The former held that any system of empirical laws[15] that led to suitably unique representations would be equally good, and subsequently Stevens (1946, 1951) classified measurement into the now famous scheme according to degree of uniqueness: nominal (for classification), ordinal (strictly increasing transformations), interval (positive power transformations), and ratio (similarity transformations).[16,17] Unfortunately, the then lack of any real examples of Stevens' contentions coupled with the vagueness of the psychologists' discussions proved unconvincing to the ad hoc committee of the British Association for the Advancement of Science (Ferguson *et al.*, 1940); they pronounced extensive structures to be the soul source of fundamental measurement. Supporting examples for Stevens' view of structures with numerical representations but no directly observable associative operation, such as intensive and conjoint ones, were only subsequently discovered.

[13] An *automorphism* of a structure is an isomorphism of the structure onto itself.

[14] In fact, as we shall see shortly, only the quite regular ones have nontrivial automorphism groups.

[15] Stevens often spoke of *rules* rather than laws, but that has a prescriptive flavor which does not capture well the scientific enterprise which speaks of "laws of nature."

[16] These statements are correct when the representation is into the positive reals. For representations into the entire reals, interval scales involve positive affine transformations and ratio scales take the form of difference transformations.

[17] I heard it said in Cambridge that this classification arose from discussions of an interdisciplinary faculty seminar on the philosophy of science that involved some prominent Cambridge mathematicians, physicists, and philosophers of the era. I am not sure there is any documentation supporting this, certainly plausible, claim.

Even so, it remained totally opaque as to why Stevens' scheme of scale types was so limited. Was that an accident of selection or did something deeper underlie it? Narens (1981 a, b) addressed this question explicitly in the context of any structure that can be mapped *onto* the reals. He defined two concepts: *M-point homogeneity*[18] for structures whose automorphism group was sufficiently rich that any ordered set of M distinct elements could be mapped by an automorphism into any other such ordered set; and *N-point uniqueness*, which is said to hold when any automorphism with N or more fixed points must be the identity. In infinite structures, $M \leqslant N$. The automorphism group (and sometimes the structure giving rise to that group) is said to be of scale type (M, N) provided M is the maximum degree of homogeneity and N is the minimum degree of uniqueness. So, for example, extensive structures with ratio-scale, additive representations are of scale type $(1, 1)$, whereas conjoint ones with interval-scale, additive representations are of type $(2, 2)$. He also defined an asymptotic order, $\succsim'$, on the automorphism group which, in the N-point unique case, is a total order.

Under the restrictions that $1 \leqslant M = N < \infty$ and that the domain of the structure be the real continuum, Narens proved that $M = N \leqslant 2$. Theodore Alper, an undergraduate major in mathematics at Harvard, encountered this work in my seminar, and he completed under the direction of Andrew Gleason an honors thesis on it that solved the general case showing that the only possible scale types are $(1, 1)$, $(1, 2)$, and $(2, 2)$. Moreover, Alper established that any such structure has a representation onto the positive reals in which the automorphisms form a subgroup of the group of positive powers. In particular, the ratio or $(1, 1)$ case consists of all transformations $x \to rx$, $r > 0$; the interval or $(2, 2)$ case, all positive power transformations $x \to rx^s$, $r > 0$, $s > 0$; and the $(1, 2)$ case is in between, e.g., the special power transformations with s restricted just to numbers of the form k^n, where k is fixed and positive and n varies over the integers. This work appeared as Alper (1987), for which he later was honored by the Young Investigator Award of the Society for Mathematical Psychology.

The basis of Alper's proof was to show that the *translations*—the identity plus all automorphisms with no fixed point—form a homogeneous Archimedean ordered group. Thus, by Hölder's theorem, they can be mapped onto the multiplicative positive reals and, by homogeneity, the structure itself can be mapped into its automorphism group, thereby inducing a map of the structure into the positive reals in such a way that the translations appear as multiplication.

[18] M-point homogeneity is very similar to the idea of M-transitivity in the theory of permutation groups, the main (and important) difference being that the former requires the order of elements to be preserved.

Freyd's comment about measurement theory being little more than applications of Hölder's theorem continues to hold, not at the structural level, but rather at the level of automorphism groups. In this case, at least, the application is decidedly nontrivial.

Subsequent work has focused on several issues: How does the theorem change when other structural assumptions replace the strong one that the structure is onto the reals? In particular, exactly what lies behind each of the key properties of the translations? What happens when we examine nonhomogeneous structures? What happens when structures are not finitely unique? How can the idea of distributive triples (Section 2.4) be modified to take into account these more general, nonadditive structures? How do these developments about scale type illuminate the issues left unresolved in the foundations of dimensional analysis? And can we apply any of this new understanding to concrete behavioral science problems? Remarks on these issues form the remainder of this article.

4.3. *Properties of Translations*

As was noted, the heart of Alper's result was to use properties of the continuum, homogeneity, and finite uniqueness to prove that the translations are homogeneous, a group, and Archimedean ordered. A surprisingly difficult part of his proof is in showing that they form a group, which is equivalent to several things: (i) they are closed under composition of functions; (ii) they are 1-point unique; (iii) if $\mathscr{T}$ denotes the set of translations and $\mathscr{D}^*$ denotes the set of all other automorphisms—these have one or more fixed points and are called *dilations*—then $\mathscr{T}\mathscr{D}^* \subseteq \mathscr{D}^*$. One exercise, to which I return every now and then, is to try to understand as fully as possible just what underlies each of the conditions on $\mathscr{T}$. In particular, one would like to see how much can be obtained without invoking Dedekind completeness and to find possible structural substitutes for homogeneity.

Recently, with important help from Alper in correcting and improving the proof, the following result has been obtained (Luce, in preparation). Consider the following four properties of an ordered structure: the asymptotic order of its automorphism group is connected, the structure is homogeneous, the set of translations is Archimedean, and the set of dilations is Archimedean relative to all automorphisms. It is not difficult to show that if a homogeneous structure has a real representation in which the translations can be represented by multiplication by positive constants, then the other three conditions are satisfied. The converse is somewhat more difficult to show. The proof, like Alper's, entails using homogeneity to map the structure isomorphically into the translations, then showing that the translations form a homogeneous, Archimedean ordered group, which, using Hölder's theorem, are mapped into the positive reals in such a way that the automorphisms appear as positive power transformations. Thus, the structure has a numerical representation and is, in fact, 2-point unique. Because experience with additive structures has repeatedly shown that results based on the continuum (Dedekind completeness) can be readily reduced to theorems involving Archimedeaness, I had hoped that this new result based not on the continuum but only on Archimedeaness would lead to an easy proof of Alper's sufficient condition. So far, that reduction has eluded us.

4.4. *Scale Types of Concatenation and Conjoint Structures*

In studying general (i.e., nonadditive) concatenation and conjoint structures, the first thing to note is that the reduction of intensive structures to conjoint ones via Eq. (8) in no way depends on bisymmetry, and the reduction of conjoint structures to that of concatenation structures via Eq. (4) in no way depends on using the Thomsen condition (or double cancellation) to establish associativity in the induced concatenation structure.

Thus, to the degree we understand the induced concatenation structures, we also understand the intensive and conjoint ones. Indeed, suppose $\mathscr{C} = \langle A \times P, \succsim \rangle$ is a conjoint structure and $a_0 \in A$, $p_0 \in P$ are used to induce an equivalent concatenation structure via Eq. (4), that (α, η) is a (factorizable) order automorphism[19] of $\mathscr{C}$, and π is the mapping induced from A to P by Eq. (4a). Then, $\eta\pi = \pi^{-1}\alpha$, and α (and so η) establishes an isomorphism between the concatenation structure induced by (a_0, p_0) and that induced by $(\alpha, \eta)(a_0, p_0) = (a_0', p_0')$. So, in the homogeneous case all the induced structures are isomorphic. Further, if α is an automorphism of an intensive structure, then (α, α) is a (factorizable) order automorphism of the induced conjoint structure $\langle A \times A, \succsim' \rangle$ and for the induced concatenation structure, Eq. (4), two cases arise: If α is a translation, then the concatenation structure induced by (a_0, a_0) is isomorphic to that induced by $(\alpha(a_0), \alpha(a_0))$. If α is an automorphism with a fixed point a_0—a *dilation*—then α is an automorphism of the concatenation structure induced by (a_0, a_0). Moreover, all of the automorphisms of the induced structure have a_0 as a fixed point. Note that the latter structure is not homogeneous because of a_0, but it is homogeneous on either side of a_0 if the conjoint structure is suitably homogeneous. This type of structure is taken up in Section 4.5. Some of these developments just mentioned are in Luce and Cohen (1983) and others are in Luce and Narens (1985).

[19] Technically, such an automorphism is called *factorizable* because it entails separate transformations of the two components. In general there are order automorphisms that are not factorizable in this sense.

Luce and Narens (1985) also explored in detail homogeneous, finitely unique concatenation structures. We showed that such structures must be at most 2-point unique; that they have the representation of Eq. (24); that the (1, 2) and (2, 2) cases must be idempotent ($a \circ a \sim a$); and that these cases impose constraints on the function f of the representation. Indeed, in the case of the continuum, the (2, 2) or interval case is of the form

$$x \oplus y = \begin{cases} x^c y^{1-c}, & x \geqslant y, \\ x^d y^{1-d}, & x < y, \end{cases} \tag{25}$$

with the two parameters $c, d \in (0, 1)$. This representation, when transformed logarithmically onto the reals, is the same as the rank dependent one of utility, Eq. (9). This form was first suggested in a psychological context by Birnbaum, Parducci, and Gifford (1971), who called it a range model.

It is worth noting that many social scientists believe that it is easier to work with ordinal scales rather than interval ones and with interval scales rather than ratio ones because the sequence of scales is increasingly restrictive: ratio is a "stronger" measurement than interval which in turn is "stronger" than ordinal. It certainly is easier to collect ordinal data than interval, and it may be easier to collect interval data than ratio. But for theory construction exactly the opposite order of strength is the case. The theory is more constrained, not less, by the "weaker" of two scales. For example, the condition of being a ratio scale admits any suitably monotonic function f in Eq. (24) whereas the weaker interval scale reduces it to the 2-parameter case of Eq. (25), which is a far more restrictive theory.

So far no one has looked seriously at the possible generalizations of the utility theory of gambles and their joint receipt using the general form of Eq. (24) rather than the quite restrictive Eq. (25).

4.5. *Structures with Singular Points*

Alper (1987) characterized not only the automorphism groups of homogeneous, finitely unique structures, but also those of nonhomogeneous, finitely unique ones. The latter groups are, in most cases, exceedingly complex, and to my knowledge no one has used his results. The problem is that nonhomogeneity stands to homogeneity as nonlinearity does to linearity: highly nonspecific to highly specific. As was noted above, some nonhomogeneous structures arise naturally in reducing conjoint structures to concatenation ones. Happily, however, they are nearly homogeneous; the trouble is limited to one point. There are other examples. If one does not leave out the null object in an extensive structure, it too becomes nonhomogeneous, but just at that null point. Another example is relativistic velocity which is also homogeneous everywhere except at velocity 0, which has the unique property that $u \oplus 0 = u$, and at the speed of light, c, which has the unique property that $u \oplus c = c$ [see Eq. (2)]. And Section 3 examined in some detail utility structures that are homogeneous on either side of the status quo.

Motivated by such examples I explored the following types of structures (Luce, 1992a). There is a general operation, i.e., function into the domain, involving finitely many arguments from the domain, and the structure is finitely unique. A point is said to be *singular* if it is fixed under all automorphisms. The structure is assumed to be homogeneous between any pair of adjacent singular points, and the operation is assumed to be monotonic in the normal fashion at nonsingular points and in a slightly modified fashion at singular ones. One then shows that there can be at most three singular points: minimal, maximal, and interior. Their possible properties are established; in particular, the interior one acts somewhat like a multiplicative unit in the nonnegative, multiplicative reals, the maximum somewhat like infinity, and the minimum somewhat like zero. One uses the known representations for homogeneous structures on the continuum to patch together a representation in which the translations—now generalized to be automorphisms with no fixed points other than the singular ones—take the following form: There is a constant c of the structure such that each translation corresponds to some $r > 0$ with $x \to rx$, for x above the interior point, and $x \to r^c x$ for x below the interior point.

4.6. *Nonfinitely Unique Structures*

As with nonhomogeneity, very little is understood in general about structures that are not finitely unique. The most obvious case is continuous, homogeneous ordinal structures in which strictly increasing functions form the automorphisms. For any positive integer N, it is possible to find two increasing functions that agree at N points. More interesting is the question of structures with more structure than ordinal that still are not finitely unique. A case in point, studied by Narens (1994), are dense threshold structures. Perhaps the simplest case is $\langle \mathbb{R}, \geqslant, T \rangle$ where $\mathbb{R}$ denotes the real numbers and T is defined by $T(x) = x + 1$, $x \in \mathbb{R}$. The interpretation is that y is seen as "discriminably larger" than x when $y > T(x)$ and x and y are "indiscriminable" when $|x - y| \leqslant 1$.

Consider the following transformations. Let $\mathscr{A}$ denote all strictly increasing functions from $(0, 1]$ onto $(0, 1]$. For each $r \in \mathbb{R}$, $\alpha \in \mathscr{A}$, integer m, and each $x \in (m, m+1)$, define

$$\alpha_r(x) = m + r + \alpha(x - m).$$

Let $\mathscr{G}$ denote the set of all these transformations generated from $r \in \mathbb{R}$ and $\alpha \in \mathscr{A}$. Narens (1994) proved, among many other things, that $\mathscr{G}$ is the automorphism group of $\langle \mathbb{R}, \geqslant, T \rangle$. Quite clearly, $\mathscr{G}$ is at least 1-point homogeneous and it is not N-point unique for any finite N.

No general results are known about the class (M, ∞) of structures. We do not even know if examples exist for every positive integer M. Presumably these classes are too diverse to expect simple characterizations, but there may well be a number of interesting subclasses, much as there is for nonhomogeneity.

4.7. *Generalized Distributive Triples*

Because distributive triples lie at the basis of dimensional analysis, a natural question to ask is whether more general ratio scale structures can substitute for extensive structures in constructing the space of attributes. If so, then the scope of physical-like measures can, in principle at least, be extended to nonextensive, ratio scales. This possibility seems potentially significant for the biological and behavioral sciences.

The question, then, was how to generalize the concept of distribution to structures with no concatenation operation. This turned out not to be difficult (Luce, 1987). Suppose $\mathscr{C} = \langle A \times P, \succsim \rangle$ is a conjoint structure. Two n-tuples $(a_1, ..., a_n)$ and $(b_1, ..., b_n)$ from A are said to be *similar* if for some $p, q \in P$, $(a_i, p) \sim (b_i, q)$ for $i = 1, ..., n$. A relational structure $\mathscr{A} = \langle A, \succsim_A, S_1, ..., S_k \rangle$, where $\succsim_A$ is the order on A induced by $\succsim$ when the P component is held fixed, is said to *distribute* in $\mathscr{C}$ if, for each j, whenever two n-tuples are similar and one is in S_j, then so is the other. Distribution of an extensive structure in an additive conjoint one (Eq. 5) is a special case of this definition. Suppose that a general structure $\mathscr{A}$ distributes in the concatenation structure $\mathscr{C}$; that the automorphism group of $\mathscr{A}$ is homogeneous and Archimedean ordered, and so has a unit representation[20] ϕ_A; and that solvability holds in $\mathscr{C}$. Then one can show that there is a mapping ϕ_P from P to the reals, such that $\phi_A \phi_P$ represents $\mathscr{C}$. Note that the "additivity" of $\mathscr{C}$ is not assumed; it is a consequence of the other conditions.

One conclusion from this line of work is that dimensional analysis can be extended to incorporate any structure that has a ratio scale representation and that distributes in a suitable conjoint structure.

In passing it is worth noting that the distribution result just mentioned rests very much upon the fact that each automorphism of $\mathscr{A}$ forms one factor of a factorizable automorphism of the conjoint structure $\mathscr{C}$. I suspect that for the nondistributive interlock that some bounded concatenation representations exhibit with conjoint structures, the nonfactorizable order automorphisms of $\mathscr{C}$ play a critical role. That is certainly the case in relativistic velocity where the Lorentz transformations are the appropriate ones, and they are not factorizable. This observation has not yet been successfully pursued in a general fashion.

[20] A unit representation is defined to be one whose automorphisms are multiplication by positive constants.

4.8 *Meaningfulness and Dimensional Analysis*

As was noted in Section 2.5, the FM-1 presentation of dimensional analysis in 1971 was incomplete because it provided no real explanation as to why physical laws should satisfy the condition known as dimensional invariance, which says that a physical law is invariant under the class of numerical transformations called *similarities*. The arguments provided for accepting dimensional invariance were not, in our opinion, very convincing. Luce (1978) (see also Section 22.7 of FM-3) showed that if one carries out the construction of the space of "physical" attributes using distributive triples, then the similarities are nothing but the factorizable automorphisms of the structure. And it had long been argued that if a relation, which is what a physical law is, is definable in any reasonable sense within a structure, then it must be invariant under the automorphisms of the structure. This was the stance of Klein (1878/1893) concerning the concept of geometric object, and it is a reasonable interpretation of Stevens' (1946, 1951) strictures on admissible statistics. Mundy (1986) explored these issues very carefully in an effort to place the whole discussion on a sounder philosophical footing.

Dzhafarov (1995) has criticized this view of dimensional invariance, arguing that traditional treatments, such as Sedov's (1956/1959), are quite satisfactory and that one need only invoke the concept of truth, not invariance under automorphisms. Despite numerous lengthy conversations and much correspondence, neither of us has persuaded the other of his position.

Although such invariance is a necessary condition for definability, which in the present context has come to be called the *meaningfulness* of a relation within a relational system, it has bite only when the automorphism group is very rich, i.e., when the structure exhibits a good deal of homogeneity or, as the physicists say, symmetry. It becomes trivially unrestrictive and so uninteresting in cases where the identity is the only automorphism, as is true in highly inhomogeneous structures. In addition to Mundy (1986), Narens (in preparation) accepted the very deep challenge involved in understanding this problem and he has treated it in a fully axiomatic, logical fashion by extending the Zermelo–Fraenkel axiom system for set theory in two ways: adding a primitive of meaningfulness and a partition of the underlying domain into purely mathematical objects and nonmathematical ones. He has explored a complex set of interlocking axioms that greatly illuminate the relation between the concept of meaningfulness and invariance under various groups of transformations.

I have read large portions of his manuscript and I believe that it will come to be considered one of the landmarks of late 20th century philosophical thought when it finally appears.

5. APPLICATIONS OF CONTEMPORARY MEASUREMENT IDEAS

The ideas described in Section 4 are relatively new and only a few applications have yet been published. Part of the reason for so few is that many of the new results simply clarify what is possible in the way of measurement. But some are far more specific. For example, Eq. (24) for a general ratio scale concatenation structure is quite specific, but to my knowledge it has yet to be used. Most of the applications involve the concept of meaningfulness and transformations under automorphisms. A number of such applications are incorporated in Narens book, but since they are not generally available, I will not attempt to describe them in any detail. A good published source for many earlier theoretical applications to psychophysics of both classical and contemporary measurement is Falmagne's 1985 book *Psychophysical Theory*. There one finds illustrated some of the major themes of such theoretical work: the use of invariance and meaningfulness arguments to limit the form of laws and the widespread use of the mathematical technique of functional equations which I mentioned in Section 3.3. Here I shall limit myself to two issues: the form of psychophysical laws and the merging of scales from different sources.

5.1. *Psychophysical Laws*

Consider cross-modal matching where to each stimulus in one ordered domain the subject assigns a stimulus from another ordered domain, e.g., sound intensity to light intensity. In a sense, matching also encompasses magnitude estimation and production where one of the domains is the real numbers. These three empirical methods were introduced by S. S. Stevens and used extensively by him and others (Stevens, 1975). Stevens' major conclusion from his data on matching was that the functional relation established between the two modalities is, when formulated in the usual ratio-scale physical measures, a power function. True, there were all sorts of caveats, but this was a major thrust of his position. When I first learned of this from him and gradually overcame my skepticism about the experimental method itself, which indeed is peculiar, I sensed that the finding was somewhat similar to the products of powers relations one encountered in dimensional analysis. Not having at the time a very deep understanding of dimensional analysis—that did not come until the work leading up to FM-1—I argued (Luce, 1959b) that the empirical result followed from the kind of invariance principle used in dimensional analysis. I was uneasy about the resulting conclusions, which were perfectly correct mathematically, because they seemed to say there was no content in the empirical finding beyond that of the physical units, which did not seem quite right. Rozeboom (1962) criticized what

I had done and in Luce (1962) I pretty much acknowledged that there was something wrong with my interpretation. Despite that, a follow-up literature developed that pursued and generalized the idea (Aczél, Roberts, & Rosenbaum, 1986; Falmagne & Narens, 1983, Krantz, 1972)

With the insights we had obtained about ratio scales, I returned to the problem in Luce (1990) where I made the following observation. Let M denote the matching relation between two modalities, so that if s is a stimulus in modality 2 that a subject matches to stimulus x in modality 1, we write xMs. The observation was that the psychophysical law M is, in a sense, *compatible* with the physics of the two attributes being matched if the following holds: for each translation τ of modality 1 there is a translation σ_τ of modality 2 such that for all x in modality 1 and all s in modality 2

$$xMs \qquad \text{if and only if} \qquad \tau(x)\, M\sigma_\tau(s). \tag{26}$$

It is not difficult to show that this holds if and only if for the ratio scale representation ϕ_1 of modality 1 and ϕ_2 of modality 2, there are positive constants α and ρ such that

$$\phi_2(s) = \alpha\phi_1^\rho(x). \tag{27}$$

This is a far more satisfactory formulation than that of Luce (1959b) in that Eq. (26) is clearly a substantive assumption, one that can be proved wrong, whereas the earlier argument had a certain tautological quality about it. The reason is that, as dimensional analysis was then formulated by the physics and engineering communities, the notation did not distinguish very clearly between automorphisms and changes of unit—both being represented by multiplication by a positive constant. There is, of course, an enormous difference. The unit is a convention of the scientist about representing the dimension, and changing it is merely a numerical matter; whereas an automorphic change, as in Eq. (26), is a systematic shift in the stimuli themselves, not in their representation. This confounding has plagued many otherwise good discussions of dimensional analysis, and for a long time blurred the real meaning of the concept of scale type.

Narens and Mausfeld (1992) have pursued these issues further, asking in such situations which parameters of laws, e.g., the exponent of Stevens' power law or Weber's constant, can be viewed as meaningful when alternative, equally good, axiomatizations of the physics using different primitives are considered. This leads to such conclusions as the following: If $\phi(a_1)$ and $\phi(a_2)$ are the usual physical measures of a pair of stimuli that are "just noticeably different", then the Weber fraction $(\phi(a_2) - \phi(a_1)/\phi(a_1)$ is not really suitable to be compared across modalities because of a lack of suitable invariance under alternative physical representations; whereas the closely related $\phi(a_2)/\phi(a_1)$ is. And Narens (in press) has given a subtle analysis of magnitude estimation in which he considers very carefully the role and meaning of the numbers arising from the method. This is generalized in his book as an abstract formulation of the cognitive situation faced by subjects, and he shows why power functions arise in a very natural fashion. It is difficult to describe these interesting contributions briefly, and I do not attempt to do so here.

5.2. *Merging Functions*

A second problem, related to, but significantly different from, the preceding one can be traced back, in part, to Arrow (1951) and to Luce (1959b, 1964). It is the familiar normative social issue of merging the views of a number of people into some sort of a social consensus. The social choice literature stemming from Arrow's work typically assumes that each individual provides a simple ordering of a set of alternatives and the goal is find a function (rule) that maps these into a social ordering or into a set of socially acceptable choices. Imposed on the rule are properties that are viewed as normative conditions of social fairness. This can be viewed as an ordinal version of the social consensus problem. Arrow's striking result was that a set of conditions, each of which seems eminently fair, is in fact inconsistent. This disturbing result has resulted in extensive work concerning alternative formulations that in some sense avoid the dilemma.

A second literature, stimulated by somewhat different considerations, has arisen independently of the social choice problem. Typically, it begins with numerically scaled inputs—ratings of some sort—from the individuals and asks questions about how best to merge them into a social rating for each of the alternatives. Here one works with numbers and not just orderings. We may describe the problem as follows: Suppose $f_1, f_2, ..., f_n$ are the individual rating functions, i.e., from a domain A of entities being rated into the real numbers. Thus, for $a \in A$, $f_1(a)$ is the rating individual i assigns to entity a. Then a merging rule F is such that for $a \in A$,

$$F(f_1, f_2, ..., f_n)(a) = F[f_1(a), f_2(a), ..., f_n(a)]. \tag{28}$$

Two very well known and important merging functions are, of course, the arithmetic mean

$$A(f_1, f_2, ..., f_n)(a) = \frac{1}{n}\sum_{i=1}^{n} f_i(a), \tag{29}$$

and the geometric mean

$$G(f_1, f_2, ..., f_n)(a) = \left(\prod_{i=1}^{n} f_i(a)\right)^{1/n}. \tag{30}$$

The question is when to apply which and what others should be considered. Here, the predominant issue is which merging functions F are *meaningful* (in the technical sense of invariance, Section 4.8) given the scale types of the inputs to F and the scale type of its output.

The basic functional equation can be formulated as follows. Suppose that T_i is a transformation of the data f_i from individual i that is admissible under its scale type, then we expect these changes to give rise to an admissible transformation $D(T_1, T_2, ..., T_n)$ so that Eq. (28) yields the invariance condition:

$$F[T_1(f_1), T_2(f_2), ..., T_n(f_n)] = D(T_1, T_2, ..., T_n)[F(f_1, f_2, ..., f_n)]. \quad (31)$$

The literature explores the numerous possible special cases depending upon the scale types involved. Of course, as Alper's theorem (Section 4.2) makes clear, for many purposes we can restrict ourselves to ratio, interval, and ordinal types. (The possibilities involving (1, 2) scale types have been ignored in large part because we do not have any serious examples of them.) Even so, there are many possibilities, some of which are: the T_i's are of the same type, but are independent; for all i, j, $T_i = T_j$; the T_i are interval scales with the same unit but different zeros; some of the T_i are of one scale type and others are of a different one; etc. Very many of these cases have been worked out in mathematical detail by Aczél and Roberts (1989), Aczél, Roberts, and Rosenbaum (1986), Luce (1964), and Osborne (1970). I will summarize only a few of the simpler, but possibly more useful, results.

We say that F is *idempotent*[21] if for each x in the domain of F, $F(x, x, ..., x) = x$. This says if everyone agrees on the evaluation x, then the merged evaluation should be x. We say that F is *symmetric* if it is invariant under any permutation of the arguments. This says that the individuals are indistinguishable and their roles can be interchanged without affecting the consequence. Although both properties seem plausible, one can wonder if they really are when, for example, different scale types apply to different people.

Aczél and Roberts (1989, Corollary 3.1, p. 236) show among many other things that if the f_i are all ratio scales with independent units and F is symmetric and idempotent, then the assertion, for any entities a and b and any number $\alpha > 0$,

$$F[f_1(a), f_2(a), ..., f_n(a)] = \alpha F[f_1(b), f_2(b), ..., f_n(b)] \quad (32)$$

is meaningful (invariant under ratio scale transformation) if and only if $F = G$, the geometric mean (Eq. 30). On the other hand, if the f_i are all interval scales and F is idempotent, then Eq. (32) is never meaningful (Corollary 3.5, p. 238).

[21] A number of other terms are also used: reflexive, agreement, and identity.

Consider the following type of assertion: for any $\alpha > 0$ and any $a, b, c, d \in A$,

$$F[f_1(a), f_2(a), ..., f_n(a)] - F[f_1(b), f_2(b), ..., f_n(b)] = \alpha(F[f_1(c), f_2(c), ..., f_n(c)] - F[f_1(d), f_2(d), ..., f_n(d)]), \quad (33)$$

and

$$F[f_1(a), f_2(a), ..., f_n(a)] > F[f_1(b), f_2(b), ..., f_n(b)]. \quad (34)$$

If the f_i are ratio scales with independent units and F is idempotent and symmetric, then Eqs. (33) and (34) are meaningful if and only if $F = G$ (Corollary 3.3, p. 238). In the interval case with F idempotent and symmetric, Eqs. (33) and (34) are never meaningful if the f_i are independent interval scales, but when these interval scales have the same unit but independent zeros these equations are meaningful if and only if $F = A$, the arithmetic mean (Eq. 29) (Corollary 3.6, p. 238).

When we turn to the ordinal cases, we begin to have some overlap with the social choice literature. I will mention only one result here, due to Ovchinnikov (in press), from the meaningfulness approach. A merging function is called a *mean value* if it is symmetric and

$$\min\{x_1, x_2, ..., x_n\} \leqslant F(x_1, x_2, ..., x_n) \leqslant \max\{x_1, x_2, ..., x_n\}. \quad (35)$$

Clearly F is idempotent. Examples are the minimum, maximum, and median. If the T_i are all the same ordinal transformation T and if D in Eq. (31) is also T, one calls F *ordinally stable*. Ovchinnikov shows that if the domain of arguments has no gaps and the ordinal transformations are homogeneous (Section 4.2), then any continuous F that is ordinally stable is an order statistic.

6. CONCLUDING REMARKS

Although the representational theory of measurement began largely as a way to understand the source of physical measurement, as described in Sections 2.1–2.3, in the second half of this century it has increasingly served two additional roles. It has greatly generalized our understanding of the scope of measurement possibilities (see Sections 2.4–2.6 and 4) and it has, increasingly, provided useful applications to the behavioral and social sciences (Sections 2.3, 2.6, 3, and 5). I believe that as the newer ideas and results become better known, additional contributions to substantive areas in the behavioral sciences will increase substantially. To the degree that behavioral theory entails both measurement and interlocking laws, such developments are inevitable—the only question is the speed with which they are carried out.

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Received: January 1, 1995