Meaningfulness and the Possible Psychophysical Laws

Perceived Risk

Pollatsek and Tversky (1970)

(a, p, b), where a, b, and p are real numbers and $0 < p < 1$ stands for the simple gamble of receiving a dollars with probability p, and b dollars with probability $(1 - p)$.

$R(g)$, for a simple gamble g, stands for the subject's perceived risk of of g.

Derived Mathematical Model

Pollatsek and Tversky's model says that

$$R(g) = tV(g) - (1-t)E(g)\,, \tag{1}$$

where $E(g)$ is the expectation of g, $V(g)$ is the variance of g, and $0 < t < 1$ is an individual parameter that varies from subject to subject.

The Model's Intuitive Appeal

The intuitive appeal of the model is that

(i) the perceived risk of simple gambles increase with variance and decline with positive expectation,

(ii) individuals vary in their risk perceptions according to how they trade-off increased variance with increase expectation, and

(iii) the individual variability is captured by the single parameter t.

Because of the relationship of (**iii**) to (**i**) and (**ii**), the size of t apparently has a psychological interpretation. It will be argued that this is not the case.

Roskam's Criticism (1989)

Relabel the simple gamble (a, p, b) as $(\$a, p, \$b)$ in order to make explicit that the outcomes are paid in dollars.

The notation $(f\ c,\ q,\ f\ d)$ will stand for the gamble of receiving c guilders with probability q and d guilders with probability $(1 - q)$.

Roskam's Gambles

Assume the subject's value of t is .7, and the exchange rate is f 2.56 to \$1. Consider the gambles given in Table 6.

	gamble	$E(g)$	$V(g)$	t	$R(g)$
g_1	(\$1,.5,-\$.50)	+0.25	0.5625	.7	0.31875
g_2	(f 2.56,.5,-f 1.28)	+0.64	3.6864	.7	2.3885
g_3	(\$0,.5,-\$1.25)	-0.625	0.3906	.7	0.4609
g_4	(f 0.0,.5,-f 3.20)	-1.6	2.56	.7	2.272

Table : Roskam's gambles

Note that at the exchange rate of f 2.56 to \$1,
g_2 is a translation of g_1, and g_4 is a translation of g_3,
and that the values $R(g_2)$ and $R(g_4)$ are computed by applying Equation 1 directly to the guilder amounts.

A Meaningfulness Issue

Table 6 displays an inconsistency: Because $R(g_1) < R(g_3)$, the subject should perceive g_1 as less risky than g_3. Therefore, because g_2 is a translation of g_1 and g_4 is a translation of g_3, it follows by assumption that the subject should perceive g_2 as less risky as g_4.

However, in Table 6, $R(g_2) > R(g_4)$, which by the model yields the subject should perceive g_2 as more risky than g_4.

There is clearly an inconsistency here.

Averaging of Rating Data

Suppose:

- R_1, R_2, and R_3 are three people involved in rating two performances, A and B.
- For $i = 1, \ldots, 3$, R_i uses the measuring function φ_i from a ratio scale $\mathcal{S}_i$ to rate A and B.
- A is judged at least as good B, in symbols, $B \preceq A$, if and only if the arithmetic mean of the ratings from R_1, R_2, and R_3 for A is greater than or equal to the arithmetic mean of the ratings for B; otherwise B is judged at least as good A, in symbols, $A \preceq B$.

Averaging of Rating Data (2)

Suppose further that the following data have been collected:

$$\begin{array}{lll} \varphi_1(A) = 5 & \varphi_2(A) = 5 & \varphi_3(A) = 5 \\ \varphi_1(B) = 8 & \varphi_2(B) = 9 & \varphi_3(B) = 1\,. \end{array}$$

The arithmetic means of A's and B's ratings are respectively 5 and 6. Thus by applying the arithmetic mean to these scores,
Scoring 1: $A \prec B$.

However, to justify this conclusion—and the rule from which it is derived—more needs to be said about the selection of the φ_i from $\mathcal{S}_i$.

This is where a meaningfulness issue involving arithmetic averaging enters.

Averaging of Rating Data (3)

$$\varphi_1(A) = 5 \qquad \varphi_2(A) = 5 \qquad 2\varphi_3(A) = 10$$
$$\varphi_1(B) = 8 \qquad \varphi_2(B) = 9 \qquad 2\varphi_3(B) = 2\,.$$

By applying the arithmetic mean to these scores,
Scoring 2: $B \prec A$, contradicting Scoring 1: $A \prec B$.

This sort of inconsistency makes the following conclusion unavoidable: If the arithmetic mean ranking rule is to be valid in the above situation with the $\mathcal{S}_i$ being ratio scales, then the selection of measuring functions from the $\mathcal{S}_i$ must be coordinated.

The problem of interpersonal comparisons of utilities.

Averaging of Rating Data (4)

Without validly being able to intercompare values, the arithmetic mean looses most of its intuitive appeal as a plausible statistic for determining overall rankings.

This suggests looking at other rules for producing overall rankings for cases where the raters' measuring functions are uncoordinated.

Averaging of Rating Data (5)

Suppose D is just as good as C if and only if the geometric mean of the ratings d_i for D is $\geq$ the geometric mean of the ratings c_i for C.

Here it is assumed that that there n raters, $i = 1, \ldots, n$, and c_i and d_i come from a measuring function from rater i's ratio scale.

Averaging of Rating Data (6)

Because for positive real numbers $c_1, \ldots, c_n$, $d_1, \ldots, d_n$, and $r_1, \ldots, r_n$,

$$(c_1 \cdot c_2 \cdots c_n)^{\frac{1}{n}} \;\geq\; (d_1 \cdot d_2 \cdots d_n)^{\frac{1}{n}}$$

$$\text{iff}$$

$$(r_1 c_1 \cdot r_2 c_2 \cdots r_n c_n)^{\frac{1}{n}} \;\geq\; (r_1 d_1 \cdot r_2 d_2 \cdots r_n d_n)^{\frac{1}{n}}\,,$$

it is easy to see that the geometric mean rule is **meaningful** in the sense that it produces the same ordering between C and D no matter which measuring functions are chosen from i's ratio scale, $i = 1, \ldots, n$.

If this kind of rating situation is fully axiomatized, it can be shown that any meaningful rule would always produce the same ordering as the geometric mean.

Quantitative Meaningfulness

$\langle X, Q_j \rangle_{j \in J}$ is a qualitative structure, $\mathcal{S}$ is a family of isomorphisms onto the numerical structure $\langle N, T_j \rangle_{j \in J}$, and $\varphi \in \mathcal{S}$.

A relation T on N is said to be quantitatively $\mathcal{S}$-meaningful if and only if for all ψ and θ in $\mathcal{S}$ and all $x_1, \ldots, x_n$ in X,

$$T[\psi(x_1), \ldots, \psi(x_n)] \text{ iff } T[\theta(x_1), \ldots, \theta(x_n)] .$$

Symmetry Meaningfulness

$\mathfrak{X} = \langle X, Q_j \rangle_{j \in J}$ is a qualitative structure, $\mathcal{S}$ is a family of isomorphisms onto the numerical structure $\langle N, T_j \rangle_{j \in J}$, and $\varphi \in \mathcal{S}$.

Then for each ψ and θ in $\mathcal{S}$, $\psi \circ \theta$ is a symmetry of $\mathfrak{X}$ (i.e., is an isomorphism of $\mathfrak{X}$ onto itself), and for each φ in $\mathcal{S}$,

$$\mathcal{S} = \{\varphi \circ \alpha \,|\, \alpha \text{ is a symmetry of } \mathfrak{X}\} .$$

An n-relation R on X is said to be symmetry meaningful if and only if for all symmetries α of $\mathfrak{X}$ and all $x_1, \ldots, x_n$ in X,

$$R(x_1, \ldots, x_n) \;\text{ iff }\; R[\alpha(x_1), \ldots, \alpha(x_n) .$$

Immediate Theorem: Let φ be in $\mathcal{S}$ and T be a relation on N. Then T is quantitatively meaningful if and only if $\varphi^{-1}(T)$ is symmetry meaningful.

Transformation Groups

Let Y be a nonempty set. $\langle H, \circ \rangle$ is said to be a transformation group on Y if and only if H is a set of one-to-one functions from Y onto Y and $\langle H, \circ \rangle$ is a group.

Let R a n-ary relation, $n \geq 1$, on Y. Then R is said to be an invariant of H or H-invariant if and only if for all $y_1, \ldots, y_n$ in Y and all h in H,

$$R(y_1, \ldots, y_n) \text{ iff } R(h(y_1), \ldots, h(y_n)) .$$

y in Y is H-invariant if and only if $h(y) = y$ for all h in H.

$Z \subseteq Y$. $h(Z) = \{h(z) \mid z \in Z\}$. Z is H-invariant if and only if $h(Z) = Z$ for all h in H.

Higher-order invariants: $\mathcal{W}$ is a set of subsets of Y. $\mathcal{W}$ is H-invariant if and only if for each h in H,

$$h(\mathcal{W}) = \{h(U) \mid U \in \mathcal{W}\} = \mathcal{W} .$$

Erlanger Program

Analytic and synthetic approaches to geometry.

Synthetic geometries consist of a structure of primitives,

$$\mathfrak{Y} = \langle Y, T_j \rangle_{j \in J}\,,$$

and axioms about $\mathfrak{Y}$.

The Erlanger Program identifies geometries with transformation groups: **Two geometric structures (either synthetic or analytic) are said to specify the same geometry if and only if they have isomorphic transformation groups.**

The symmetry group of $\mathfrak{Y}$, $\langle H, \circ \rangle$, consists of the set H of transformations on Y that leave the primitives of $\mathfrak{Y}$ invariant. H is a transformation group, sometimes called the transformation group of $\mathfrak{Y}$.

Erlanger Program

The axioms about the primitives are used to derive group theoretic properties that specify H up to an isomorphism.

That is, any two axiomatic structures describing the same geometry will have isomorphic symmetry groups by the Erlanger Program.

Erlanger Program — Summary

Erlanger Program: *A geometry* $\mathfrak{Y}$ *is a transformation group* $\langle H, \circ \rangle$ *on a nonempty set* Y. *The relations, 0-ary, n-ary, higher-order, that are invariant under* H, *belong to* $\mathfrak{Y}$; *those that are not invariant do not belong to* $\mathfrak{Y}$.

Two gaps in the Erlanger Program:

- **Why should invariants belong to $\mathfrak{Y}$?**
- **What about the case of $\mathfrak{Y}$ having the trivial symmetry group?** (In this case every relation belongs to $\mathfrak{Y}$ by Erlanger Program)

Cartesian Geometry

$\mathcal{E}$ = the set of ordered pairs of real numbers.

In terms cartesian coordinates, the euclidean distance function d between points (x, y) and (u, v) is defined by

$$d[(x, y), (u, v)] = \sqrt{(x - u)^2 + (y - v)^2}\,,$$

and an orientation for angles is specified so that counterclockwise from the positive abscissa to the positive ordinate is a 1 right angle.

d and this orientation specify an analytic geometry for $\mathcal{E}$.

Cartesian Invariance

The set E of transformations that leave leave d and the orientation invariant are known to be those generated by translations and rotations.

Thus, by the Erlanger Program, $\langle E, \circ \rangle$ provides a transformational description of planar euclidean geometry, $\mathcal{E}$.

There are a number of synthetic axiomatizations of $\mathcal{E}$, some with different primitives than others.

The symmetry group for the structure of primitives of each of these axiomatizations is isomorphic to $\langle E, \circ \rangle$.

Measurement Theory & The Erlanger Program

qualitative structure $\longleftrightarrow$ synthetic geometry

numerical structure $\longleftrightarrow$ analytical
geometrical structure

representation $\longleftrightarrow$ isomorphism from
a synthetic to an analytic geometry

representationally meaningful $\longleftrightarrow$ belonging
to the geometry under consideration

Concern with science and empirical matters $\longleftrightarrow$ concern
with mathematics and geometric matters

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Possible Psychophysical Laws

Luce (1959) presented a theory that was intended to generalize dimensional analysis of physics to a wider range of scientific phenomena, particularly to phenomena in psychophysics. This theory became known as "Possible Psychophysical Laws," and it generated a considerable literature.

Stevens Psychophysical Law

S. S. Stevens in a well-known article—"The Power Law"—provide a measurement theory and much data that psychological magnitudes of physical stimuli were related to physical stimuli by a power law, that is, a perceived psychological magnitude $\psi(x)$ of stimulus x whose measurement when measured on a standard physical scale is $\varphi(x)$ are related by

$$\psi(x) = c\varphi(x)^r \,, \tag{2}$$

where c and r are positive real numbers.

Luce's Derivation

$$\psi(x) = c\varphi(x)^r \,, \tag{3}$$

Because of the way the experiments were carried out, Stevens' theory of measurement required ψ to be measured by a ratio scale.

Luce made the assumption that each ratio transformation $k\varphi$ of the physical measurement led to a ratio transformation $h(k)$ of the perceived psychological measurement, and showed that this implied Equation 3.

He noted the striking relationship of this form of derivation with dimensional analysis in physics. He generalized this method to apply to other scale types.

Possible Psychophysical Laws

SCALE TYPES		Functional Equation	Comments
Independent Variable	Dependent Variable		
ratio	ratio	$u(x) = \alpha x^{\beta}$	$\beta/x; \beta/u$
ratio	interval	$u(x) = \alpha \log x + b$, $u(x) = \alpha x^{\beta} + \delta$	α/x $\beta/x; \beta/u; \delta/x$
ratio	log-interval	$u(x) = \delta e^{\alpha^{x\beta}}$	$\alpha/u; \beta/x; \beta/u; \delta/x$
ratio	ratio	$u(x) = \alpha x^{\beta}$	$\beta/x; \beta/u$
interval	ratio	**impossible**	
interval	interval	$u(x) = \alpha x + \beta$	β/x
interval	log-interval	$u(x) = \alpha e^{\beta x}$	$\alpha/x; \beta/u$
log-interval	ratio	**impossible**	
log-interval	interval	$u(x) = \alpha \log x + \beta$	α/x
log-interval	log-interval	$u(x) = \alpha x^{\beta}$	$\beta/x; \beta/u$

Table : The Possible Laws Satisfying the Principle of Theory Construction
(*Note:* The notation α/x means "α is independent of the unit of x.")

Possible Psychophysical Laws Generalized

Falmagne, J.-C. & Narens, L. (1983). Scales and meaningfulness of quantitative laws. *Synthese, 55*, 288–325.

Falmagne, J.-C. (1985). *Elements of Psychophysical Theory.* New York: Oxford University Press.

Roberts F. S. & Rosenbaum, Z. (1986). Scale type, meaningfulness, and the possible psychophysical laws. *Mathematical Social Sciences, 12*, 77–95.

Aczél, J., Roberts, F. S., & Rosenbaum, Z. (1986). On scientific laws without dimensional constants. *Journal of Mathematical Analysis and Applications, 119*, 389–416.

Aczél, J., & Roberts, F. S. (1989). On the possible merging functions. *Math. Soc. Sci., 17*, 205–243.

Falmagne, J.-C. (2004). Meaningfulness and Order-Invariance: Two Fundamental Principles for Scientific Laws. *Foundations of Physics, 34*, 1341–1348.

Dimensional Analysis

Consider a simple pendulum consisting of a ball suspended by a string. The ball is displaced and is then released.

We want to find its period.

To do this we need to list the relevant physical entities that determine the period t.

Relevant Variables

The relevant variables are:

- the mass m of the ball,
- the length d of the string,
- the gravitational constant g that describes acceleration of small masses towards the center of the earth,
- and the angle z that the pendulum makes with the vertical when the ball is released.

These, are qualitative entities. They may be viewed as elements of the domain of a relational structure $\mathfrak{X}$ that has components that can be used to measure the relevant entities, for example, an extensive structure to measure length, and extensive structure to measure mass, et cetera.

Numerical Measurments

m,d,g,z and t can be properly assigned numerical values in various ways.

We will assume that they are measured in terms of a coherent set of units, for example if d is measured in terms of the unit u and t in terms of the unit v, then g is measured in terms of the unit u/v^2.

In this application z is measured by the ratio of a measured length of arc divided by a measured radius, and is thus a real number—a "dimensionless" quantity—that does not depend on the unit in which length is measured.

Dimensional Invariance

Though the use of "physical intuition" we will assume that we have enough information to determine the qualitative entity t—that is, it is assumed that t is physically determined by m,g,d and z. In terms of functional notation this is equivalent to

$$\mathrm{t} = \mathrm{F}(\mathrm{m},\mathrm{g},\mathrm{d},\mathrm{z})\,. \tag{4}$$

A proper numerical representation for Equation 4 consists in providing (*i*) proper numerical measurements, t, m, g, d, and z, to respectively, t,m,g,d and z in terms of some coherent system of units and (*ii*) finding a numerical function F such that

$$t = F(m, g, d, z)\,. \tag{5}$$

A principle of dimensional analysis called dimensional invariance requires all other proper representations of Equation 4 to be of the form:

$$t' = F(m', g', d', \theta)\,, \tag{6}$$

where t', m', g', and d' are proper numerical measurements in some other coherent system of units.

Luce's Theorem

A theorem of Luce (1978) shows that the dimensional invariance of F is equivalent to F being invariant under the symmetries of the $\mathfrak{X}$ described previously.

Invariance of z

Equation 5: $t = F(m, g, d, z)$.

Equation 6: $t' = F(m', g', d', \theta) = F(m', g', d', z)$.

By the way it was defined, z is a real number that does not depend on which unit is used to measure length. Thus z has the same numerical value in each coherent system of units, that is, $z = \theta$.

Note that in Equations 5 and 6 the same numerical function F is used to represent the qualitative function F. z also appears in Equations 5 and 6.

Ratio Scale Measurement

Because t,m, and d are measured on ratio scales, t', m', g', and d' are related to t, m, g, and d by the following: there are positive real numbers α, β, and γ such that

$$t' = \alpha t,\ d' = \beta d,\ g' = \frac{\beta g}{\alpha^2},\ \&\quad m' = \gamma m\,. \tag{7}$$

Equation 6: $t' = F(m', g', d', z)\,,.$

Substituting Equation 7 into Equation 6 yields,

$$\alpha t = F\left[\gamma m, \frac{\beta g}{\alpha^2}, \beta d, z\right]\,. \tag{8}$$

Pendulum

$$(\textit{Equation } 8) \;\; \alpha t = F\left[\gamma m, \frac{\beta g}{\alpha^2}, \beta d, z\right] \, .$$

By dimensional invariance, Equation 8 is valid for all positive reals α, β and γ. Because the measurement of z does not vary with changes of units, Equation 8 can be rewritten as

$$\alpha t = H\left[\gamma m, \frac{\beta g}{\alpha^2}, \beta d\right] \, , \tag{9}$$

where for all positive real numbers s, x ,y, and w,

$$s = F(x, y, w, z) \;\text{ iff }\; s = H(x, y, w) \, .$$

Pendulum

Equation 9 : $\alpha t = H\left[\gamma m, \frac{\beta g}{\alpha^2}, \beta d\right]$

In Equation 9 the first argument of H, γm, can take any real value by an appropriately choosing γ while leaving the value of H, αt, unchanged. From this it follows that H does not depend on its first argument.

Thus rewrite Equation 9 as

$$\alpha t = K\left[\frac{\beta g}{\alpha^2}, \beta d\right] \, . \tag{10}$$

Choose units so that $\alpha = \sqrt{g}/\sqrt{d}$ and $\beta = 1/d$. Then by Equation 10,

$$t = \frac{\sqrt{d}}{\sqrt{g}} \cdot K(1, 1). \tag{11}$$

Pendulum

Equation 11: $\quad t = \frac{\sqrt{d}}{\sqrt{g}} \cdot K(1,1)$

It is easy to verify by inspection that when the measurements of t,g, and d are changed to another coherent system of units that an equivalent form of Equation 11 is valid with the same real constant $K(1,1)$.

If it is assumed that the measurement of g is known in one coherent set of units—and thus in all coherent set of units—then Equation 11 provides a method of calculating for each length d the measurement of the period t in terms of the measurement of d and the real number $K(1,1)$.

Pendulum

Equation 11: $t = \frac{\sqrt{d}}{\sqrt{g}} \cdot K(1,1)$

The number $K(1,1)$ completely determined by dimensional analysis. However, it can be found by experiment; that is, a particular measured length can be chosen and the period measured for that particular length and $K(1,1)$ can be computed by Equation 11.

In the above argument z is assumed to be fixed. As z varies so will Equation 11; however, Equation 11 will have the same form, because only the real number $K(1,1)$ will vary. In other words, $K(1,1)$ is a function of z and therefore of z. Thus for variable angle z, rewrite Equation 11 as

$$t = \frac{\sqrt{d}}{\sqrt{g}} N(z)\,, \tag{12}$$

where z is the (dimensionless) measurement of z, and N is a fixed real valued function.

Pendulum

Equation 12: $t = \frac{\sqrt{d}}{\sqrt{g}} N(z)$

The usual law for the period of the pendulum derived from Newton's laws is essentially the same as Equation 12, but with $N(z)$ being the function $\sin(z)$.

Thus by using Newton's laws, $N(z)$ can be completely determined without having to resort to experiment.

However,the above dimensional analysis also leads to certain completely determined laws of pendulums:

For example, if two pendulums of measured lengths (in the same length unit) d_1 and d_2 are released from the same angle with the vertical, and their resulting periods are t_1 and t_2 (measured in the same time unit), then it is immediate from Equation 12 that

$$\frac{t_1}{t_2} = \frac{\sqrt{d_1}}{\sqrt{d_2}}\,. \tag{13}$$

Pendulum

Equation 13: $\frac{t_1}{t_2} = \frac{\sqrt{d_1}}{\sqrt{d_2}}$

Note that Equation 13 does not depend on which units are used to measure distance and time and that the gravitational constant is not mentioned in its derivation.

Delian Problem

427 B.C.: Double the cubical alter of Apollo at Delos.

A number of solutions for Delian problem were suggested. They all used either special instruments (other than compass and straightedge) or special geometric curves—conditions that Plato found unacceptable:

According to Plutarch, Plato found these to be extremely distasteful and considered them to be "the mere corruption and annihilation of the one good of geometry, which was thus shamefully turning its back upon the embodied objects of pure intelligence."

Descartes

1637: Descartes geometry.

Conic sections are geometric, but Archimedes' spiral is not.

Scientific Topics

Principle 1. The domain of the scientific topic is a qualitative set X.

Principle 2. The scientific topic is determine by a structure of primitives $\langle X, Q_j \rangle_{j \in J}$, where each Q_j is a higher-order relation based on X.

Principle 3. The structure of primitives and each of its primitives belong to the scientific topic.

Principle 4. The scientific topic is closed under "scientific definition;" that is, if $b_1, \ldots, b_n$ belong to the topic and b is defined "scientifically" in terms of $b_1, \ldots, b_n$, then b belongs to the scientific topic.

Principle 5. A portion of pure mathematics can be used in scientific definitions.

Full Scientific Topic

• Structure of primitives $\langle X, Q_j \rangle_{j \in J}$

• Portion of mathematics = all of pure mathematics

• Language for scientific definition = infinitary higher-order logic (e.g. Principia Mathematica) with constant symbols for X, each primitive and each pure mathematical entity
or formulas of set theory with X being a set of atoms (non-sets) and constant symbols for X, each primitive, and each pure set.

Scientific Topic and Symmetry Invariance

$\mathfrak{X} = \langle X, Q_j \rangle_{j \in J}$ a structure of primitives.

Theorem. A higher-order relation R based on X belongs to the scientific topic determined by $\mathfrak{X}$ if and only if it is invariant under the symmetries of $\mathfrak{X}$.

Conclusion. The Erlanger Program is generalized by (*i*) weakening the language for scientific definition, or (*ii*) allowing a smaller portion of pure mathematics to be used for defining the scientific topic, or (*iii*) both (*i*) and (*ii*). In this case the theorem becomes: *If a higher-order relation R based on X belongs to the scientific topic determined by $\mathfrak{X}$ then it is invariant under the symmetries of $\mathfrak{X}$.*

A Use of the Previous Theorem

Consider the situation where by extra-scientific means (e.g., intuition, experience, etc.) a scientist is led to believe that a function $z = F(x, y)$ that he needs to describe from a subset of $X \times X$ into X is completely determined by the observable, first-order relations $R_1, \ldots, R_n$ on X.

Then it is reasonable for the scientist to proceed under the hypothesis that F belongs to the scientific content of $\mathfrak{X} = \langle X, R_1, \ldots, R_n \rangle$, which for this discussion may be taken as determined by Full Scientific Topic.

Thus the scientist assumes F has a scientific definition in terms of $\mathfrak{X}$ and its primitives. By the previous theorem, F is invariant under the symmetries of $\mathfrak{X}$.

Characterization by Symmetries

Suppose the scientist knows enough properties about $\mathfrak{X}$ and has the mathematical skill to determine the symmetry group G of $\mathfrak{X}$.

Then methods of analyses involving symmetries may be employed to provide information helpful in characterizing F. There are several methods in the literature for accomplishing this.

Epistemological Principle

Note that in the above process, scientific definability is used to justify F belonging to the appropriate topic, invariance is used as a mathematical technique to find helpful information for characterizing F, and that these two uses are connected by a theorem of mathematical logic.

Also note that the scientist's belief that F belonged to the topic generated by $\mathfrak{X}$ is extra-scientific. Therefore, the deductions based on information obtained through the above process should be either checked by experiment or be derived from accepted scientific theory and facts; i.e., they should be treated as scientific hypotheses that need corroboration.

Conclusion about the Example and Dimensional Analysis

For the purposes of science, the above process is a method of generating hypotheses and not facts: If the scientist's extra-scientific beliefs are correct, then the generated hypotheses will be facts; however, the scientist has no scientific guarantee that his beliefs are correct.