Chapter 2 Construction of Numerical Functions

2.1 REAL-VALUED FUNCTIONS ON SIMPLY ORDERED SETS

In Chapter 1, we defined a *simple order* to consist of a set with a transitive, connected, antisymmetric binary relation. Theorem 1.1 established that any finite simple order can be represented by a finite set of real numbers together with their natural ordering. Such a representation is unique up to strictly increasing transformations of Re onto itself (ordinal scale). Here we prove similar representation and uniqueness theorems for certain infinite simple orders. The corresponding results for weak orders follow immediately from those for simple orders, factoring out equivalence classes of a weak order to obtain its associated simple order.

It is easy to show that not every simple order can be represented in $\langle \mathrm{Re}, \geqslant \rangle$. Let $A = \mathrm{Re} \times \mathrm{Re}$, and define $\succsim$ on A by

$$(x, y) \succsim (x', y')$$

if and only if either

(i) $x > x'$ or
(ii) $x = x'$ and $y \geqslant y'$.

Suppose that ϕ is an isomorphism of $\langle A, \succsim \rangle$ into $\langle \mathrm{Re}, \geqslant \rangle$. For x in Re, let $\phi_1(x) = \phi(x, 1)$ and let $\phi_0(x) = \phi(x, 0)$. Since $(x, 1) \succ (x, 0)$ and ϕ is

order preserving, $\phi_1(x) > \phi_0(x)$. Since every open interval of Re contains a rational number, there is a rational $\psi(x)$ such that $\phi_1(x) > \psi(x) > \phi_0(x)$. Thus, ψ is a function from Re into the set of rational numbers, denoted Ra. Moreover, ψ is one to one, since if $x > x'$, $(x, 0) \succ (x', 1)$, so

$$\psi(x) > \phi_0(x) > \phi_1(x') > \psi(x').$$

However, it is well known that the rational numbers are countable (i.e., can be put into one-to-one correspondence with the set I^+ of positive integers) and the reals are not countable; therefore, a one-to-one mapping such as ψ cannot exist, which implies that the isomorphism ϕ also cannot exist.

The previous example shows that some additional condition must be imposed on a simple order to obtain an isomorphism into $\langle \mathrm{Re}, \geqslant \rangle$. We already know that the (structural) assumption of finiteness suffices. We next prove that it is also sufficient for the set A to be countable.

THEOREM 1. *Let $\langle A, \succsim \rangle$ be a simple order. If A is countable, then there exists a real-valued function ϕ on A such that for all $a, b \in A$,*

$$a \succsim b \quad \textit{iff} \quad \phi(a) \geqslant \phi(b).$$

Theorem 1 was proved by Cantor (1895).

Since Theorem 1 is merely a step toward a necessary and sufficient axiomatization, we do not formulate the corresponding uniqueness theorem at this point. The method of proof involves constructing ϕ precisely as was outlined in Section 1.1.1; after the values of ϕ have been found for any finite subset of A, we find the value for a new element of A by locating the new element between its nearest neighbors in the finite set already considered and assigning a number between the numbers assigned to those neighbors. Since A is countable, we construct the value of ϕ for the nth element of A at the nth step; eventually, any given element of A is reached. The function ϕ is thus considered to be defined over the whole of A, since given any particular element a in A we know precisely how to construct the number $\phi(a)$. (The function ϕ is said to be defined by *induction*.)

PROOF OF THEOREM 1. Let a_n denote the element of A that corresponds to n in the given one-to-one correspondence between A and I^+.

Define $\phi(a_1) = 0$. If $\phi(a_1), \ldots, \phi(a_n)$ have been defined, where $n \geqslant 1$, define $\phi(a_{n+1})$ as follows:

(i) If $a_{n+1} \succ a_k$ for all k, $1 \leqslant k \leqslant n$, let $\phi(a_{n+1}) = n$.
(ii) If $a_{n+1} \prec a_k$ for all k, $1 \leqslant k \leqslant n$, let $\phi(a_{n+1}) = -n$.
(iii) If neither (i) nor (ii) applies, then there exist i, j, with $1 \leqslant i, j \leqslant n$,

such that $a_i \succ a_{n+1} \succ a_j$ and for any k, $1 \leqslant k \leqslant n$, either $a_k \succsim a_i$ or $a_j \succsim a_k$. Let $\phi(a_{n+1}) = \frac{1}{2}[\phi(a_i) + \phi(a_j)]$.

By induction, ϕ is defined on all of A. To prove that ϕ is order preserving, use mathematical induction: it is obviously order preserving on $\{a_1\}$, and if it is order preserving on $\{a_1, ..., a_n\}$, then, by the construction of $\phi(a_{n+1})$ just given, it is order preserving on $\{a_1, ..., a_{n+1}\}$. (See Exercise 1.) ◇

There is a countable subset of the real numbers—the rationals—that is thoroughly interspersed in the reals in the sense that between every two reals there is a rational. If an uncountable simple order is to be represented in $\langle \mathrm{Re}, \geqslant \rangle$, then one anticipates that it will exhibit the analogous property. In fact, the trouble in the example at the beginning of the chapter is precisely that there are uncountably many intervals of form $(x, 1) \succ (x, 0)$; therefore no countable set can be found such that it has a representative in every such interval. Conversely, if an uncountable set does have a countable subset thoroughly interspersed, then by Theorem 1 we can find a representation for that subset in $\langle \mathrm{Re}, \geqslant \rangle$. We can then expect to extend the representation to the whole uncountable set by considering each element of the latter as a limit of elements in the countable subset. This suggests the feasibility of there being a simple necessary and sufficient condition for a simple order to be representable in $\langle \mathrm{Re}, \geqslant \rangle$.

To make precise the meaning of "thoroughly interspersed" we introduce the following technical concept of *order dense*:

DEFINITION 1. *Let $\langle A, \succsim \rangle$ be a simple order and let B be a subset of A. Then B is* order dense *in A iff for all $a, c \in A$ such that $a \succ c$, there exists $b \in B$ such that $a \succsim b \succsim c$.*

THEOREM 2. *If $\langle A, \succsim \rangle$ is a simple order, then the following two conditions are equivalent:*

(i) *There is a finite or countable order-dense subset of A.*

(ii) *There is an isomorphism of $\langle A, \succsim \rangle$ into $\langle \mathrm{Re}, \geqslant \rangle$.*

This theorem seems first to have been stated in this generality by Birkhoff (1948, pp. 31–32), although a result almost as strong was proved by Cantor (1895). The proof sketched by Birkhoff is incomplete. Debreu (1954, Lemma II) proved that (i) implies (ii).

For the proof of this theorem, we remind the reader of three elementary facts about countable sets:

1. *Any infinite subset of a countable set is itself countable.*
2. *If A is countable and B is finite or countable, then $A \cup B$ is countable.*

3. *The Cartesian product of finitely many countable sets is countable.*

(Since the rationals can be regarded as a subset of the Cartesian product I × I, they are countable.)

If $\langle A, \succsim \rangle$ is a simple order and if $a, a' \in A$ are such that $a' \succ a$ and, for any $b \in A$, either $b \succsim a'$ or $a \succsim b$, then we say that (a, a') is a *gap* and that a, a' are *endpoints of a gap*. With this definition, we can formulate the following lemma.

LEMMA 1. *Let $\langle A, \succsim \rangle$ be a simple order and let A^* be the set of all endpoints of gaps. If either* (i) *or* (ii) *of Theorem* 2 *holds, then A^* is either finite or countable.*

PROOF. Let A_1^* be the set of upper endpoints of gaps and A_2^* be the set of lower endpoints.

Suppose that (i) holds, and let B be a finite or countable, order-dense subset of A. By Definition 1, if (a, a') is a gap, then either $a \in B$ or $a' \in B$. Hence, $A_1^* - B$ is in one-to-one correspondence with a subset of B (each upper endpoint not in B corresponds to its lower endpoint which is in B). Therefore, $A_1^* - B$ is finite or countable. Similarly, $A_2^* - B$ is finite or countable, and we know that $A^* \cap B$ is finite or countable. So $A^* = (A_1^* - B) \cup (A_2^* - B) \cup (A^* \cap B)$ is finite or countable.

Suppose that (ii) holds, and let ϕ be an isomorphism of $\langle A, \succsim \rangle$ into $\langle \mathrm{Re}, \geqslant \rangle$. If (a, a') is a gap, then there exists a rational ρ such that $\phi(a') > \rho > \phi(a)$. This leads to a one-to-one correspondence between A_1^* and a subset of Ra and to one between A_2^* and a subset of Ra, so A^* is finite or countable. ◆

PROOF OF THEOREM 2. Suppose that B is a finite or countable order-dense subset of A. Adjoin the greatest and least elements of A (if such exist) to B. Let A^* be the set of endpoints of gaps. By Lemma 1, $B^* = B \cup A^*$ is finite or countable. By either Theorem 1.1 or 1, there exists an order-preserving function ϕ' from B^* to Re.

For $a \in A$, let $\phi(a)$ be the least upper bound of the set of numbers $\{\phi'(b) \mid b \in B^* \text{ and } a \succsim b\}$. For $a \in B^*$, obviously $\phi(a)$ exists and equals $\phi'(a)$. To show that $\phi(a)$ exists for $a \notin B^*$, note that there exist a_1, a_2 with $a_1 \succ a \succ a_2$. By order density, there exist $b_1, b_2 \in B$ such that

$$a_1 \succsim b_1 \succ a \succ b_2 \succsim a_2.$$

The set $\{\phi'(b) \mid b \in B^* \text{ and } a \succsim b\}$ is nonempty [it contains $\phi'(b_2)$] and is bounded above [by $\phi'(b_1)$], so its least upper bound $\phi(a)$ exists.

We show that ϕ is order preserving. Suppose that $a' \succ a$. By the construction of B^*, there exist $b, b' \in B^*$ such that $a' \succsim b' \succ b \succsim a$. (If $a, a' \in B^*$, let $b = a$, $b' = a'$. If $a \notin B^*$, then by order density there exists $b' \in B$ with $a' \succsim b' \succ a$. Since (a, b') is not a gap, there exists a'' with $b' \succ a'' \succ a$ and hence $b \in B$ with $a'' \succsim b \succ a$. A similar argument holds if $a' \notin B^*$.) Thus, $\phi(a') \geqslant \phi'(b') > \phi'(b) \geqslant \phi(a)$, as required.

Conversely, suppose that ϕ is an isomorphism of $\langle A, \succsim \rangle$ into $\langle \mathrm{Re}, \geqslant \rangle$. Let J be the set of pairs of rational numbers (r, r') such that for some $a \in A$, $r' > \phi(a) > r$. For each $(r, r') \in J$, choose exactly one $a \in A$ such that $r' > \phi(a) > r$, and let B_1 be the set of elements so chosen.[1] Since B_1 is in one-to-one correspondence with a subset of $\mathrm{Ra} \times \mathrm{Ra}$, B_1 is finite or countable. Let A^* be the set of endpoints of gaps. By Lemma 1, $B = A^* \cup B_1$ is finite or countable.

We show that B is order dense in A. Suppose that $a \succ c$. If (c, a) is a gap, then c, $a \in A^*$ and we can take either of them as b. Otherwise, choose b' with $a \succ b' \succ c$ and rationals r, r' with $\phi(a) > r' > \phi(b') > r > \phi(c)$. Thus, $(r, r') \in J$, so for some $b \in B_1$, $r' > \phi(b) > r$. It follows that $a \succ b \succ c$, as required. ◈

We next turn to the uniqueness theorem for ordinal measurement, which is very simple.

THEOREM 3. *Let $\langle A, \succsim \rangle$ be a simple order and let ϕ, ϕ' be two functions satisfying* (ii) *of Theorem* 2. *Let R, R' be the respective ranges of ϕ, ϕ'. Then there exists a strictly increasing function h from R to R', such that for all $a \in A$, $h[\phi(a)] = \phi'(a)$. Moreover, if h is any strictly increasing function on the range R of a representation ϕ of $\langle A, \succsim \rangle$, then $\phi'(a) = h[\phi(a)]$ defines another such representation.*

The proof is trivial, e.g., define h on R by $h[\phi(a)] = \phi'(a)$, and then show h is well defined and strictly increasing.

Note that the class of permissible transformations includes all strictly increasing functions from Re onto Re, but includes others as well. For example, there are some strictly increasing functions on subsets of Re that cannot be extended to strictly increasing functions on Re. For finite sets A, any permissible transformation on R can be extended to one from Re onto Re.

In the proof of Theorem 2, the definition of ϕ on A (i.e., as the least upper bound of $\phi'(b)$, for $b \in B^*$ and $a \succsim b$) differs in an important way from the definition of ϕ' on B^* (or of ϕ on a countable set, as in Theorem 1). Since B^* is finite or countable, there is a method of obtaining the value of ϕ' for any given element of B^* in a finite number of steps. But in order to

[1] Note the use of the Axiom of Choice. We try to point out its use, or the use of equivalents, throughout the book; we avoid using it when we know how.

obtain the value of ϕ for an element of $A - B^*$, one must first construct ϕ' for *every* element of B^*, which in the countable case requires infinitely many steps. In Section 2.2, we also use a limiting process to define ϕ, but there we have the possibility of obtaining an approximation to the value of ϕ, with any specified degree of accuracy, in a finite number of steps. The ratio-scale uniqueness theorem of Section 2.2 leads to a well-defined notion of accuracy of approximation (percentage error). With only the ordinal uniqueness of Theorem 3, however, there is no suitable notion of approximation, and so the definition of ϕ is not very satisfactory.

Finally, we note that Debreu (1954, Lemma II) has shown that the order-preserving function ϕ of Theorem 2 can always be constructed so as to be continuous in every natural topology on the simple order $\langle A, \succsim \rangle$. This is easy to prove once all the required definitions are given, but we shall not do so since the ideas involved are not used elsewhere in this book. The essence of the matter is simply this. The function ϕ will be discontinuous if there are gaps in its range, so that, for instance, $a \in A$ is the least upper bound of $A_1 \subset A$, with respect to $\succsim$, but $\phi(a)$ is greater than the least upper bound of $\{\phi(a') \mid a' \in A_1\}$. Such gaps can be closed simply by modifying ϕ to close them; for example, if there were only one gap at a, one would simply subtract the appropriate constant from all values of $\phi(a'')$ for $a'' \succsim a$. Since the set of gaps can only be finite or countable, one can easily arrange to close them all at once by subtracting appropriate sums (with a finite or countable number of terms) of constants.

Because of the relatively nonunique measurement for simple orders, we make little use of the results of this section in the remainder of the book. Theorem 2 is used in Chapters 7 and 13.

2.2 ADDITIVE FUNCTIONS ON ORDERED ALGEBRAIC STRUCTURES

We next formulate and prove several extensions of a classical theorem of Hölder (1901). The first four subsections deal with the main isomorphism theorem, Theorem 4, due to Krantz (1968), which underlies the construction of real-valued functions throughout the book. We use the weakest structural conditions now known; the cost is some slight complications in the statement of the theorem. For example, we assume that the binary operation $\circ$ is defined only for a certain set B of pairs (a, b). Intuitively, B should be thought of as the set of concatenable pairs. We do not assume that any equation can be solved, but only that if $a \succ b$ then for some c, $a \succsim b \circ c$.

Sections 2.2.5 and 2.2.7 deal with the special cases of Archimedean simply ordered groups and rings, respectively; Section 2.2.6 presents a brief comparison with other versions of Hölder's Theorem. A much more detailed treatment of the type of ordered algebraic structures considered here can be found in Fuchs (1963) and in a later survey by Vinogradov (1969).

2.2.1 Archimedean Ordered Semigroups

We start with the basic definition.

DEFINITION 2. *Let A be a nonempty set, B and $\gtrsim$ nontrivial binary relations on A, and $\circ$ a binary operation from B into A. The quadruple $\langle A, \gtrsim, B, \circ \rangle$ is an* ordered local semigroup *iff, for all $a, b, c, d \in A$, the following five axioms are satisfied:*

1. *$\langle A, \gtrsim \rangle$ is a simple order.*
2. *If $(a, b) \in B$, $a \gtrsim c$, and $b \gtrsim d$, then $(c, d) \in B$.*
3. *If $(c, a) \in B$ and $a \gtrsim b$, then $c \circ a \gtrsim c \circ b$.*
4. *If $(a, c) \in B$ and $a \gtrsim b$, then $a \circ c \gtrsim b \circ c$.*
5. *$(a, b) \in B$ and $(a \circ b, c) \in B$ iff $(b, c) \in B$ and $(a, b \circ c) \in B$; and when both conditions hold $(a \circ b) \circ c = a \circ (b \circ c)$.*

For the rest of this definition, assume that $\langle A, \gtrsim, B, \circ \rangle$ is an ordered local semigroup (Axioms 1–5 hold).

$\langle A, \gtrsim, B, \circ \rangle$ is called positive *iff, for all $a, b \in A$,*

6. *If $(a, b) \in B$, then $a \circ b > a$.*

A positive semigroup $\langle A, \gtrsim, B, \circ \rangle$ is called regular *iff, for all $a, b \in A$,*

7. *If $a > b$, then there exists $c \in A$ such that $(b, c) \in B$ and $a \gtrsim b \circ c$.*

For any $a \in A$, we define inductively a subset N_a of I^+, and we define na for each $n \in N_a$ by:

(i) *$1 \in N_a$ and $1a = a$;*

(ii) *if $n - 1 \in N_a$ and $((n-1)\,a, a) \in B$, then $n \in N_a$ and na is defined to be $((n-1)\,a) \circ a$;*

(iii) *if $n - 1 \in N_a$ and $((n-1)\,a, a) \notin B$, then for all $m \geqslant n$, $m \notin N_a$.*

(Thus, N_a is precisely the set of consecutive positive integers for which na is defined.)

$\langle A, \gtrsim, B, \circ \rangle$ is called Archimedean *iff for all $a, b \in A$:*

8. *$\{n \mid n \in N_a$ and $b > na\}$ is a finite set.*

An example of a structure satisfying Axioms 1–8 is provided by

$\langle Re^+, \geqslant, R_\Omega, + \rangle$, where $\geqslant$ is the usual order relation and $+$ the usual addition operation on Re^+, and

$$R_\Omega = \{(x, y) \mid x, y \in Re^+ \quad \text{and} \quad x + y < \Omega\},$$

where $0 < \Omega \leqslant +\infty$. That is, $+$ is defined for all pairs of positive reals whose usual sum is less than Ω. By contrast, we do not obtain an Archimedean, regular, positive, ordered local semigroup if we define $+$ for just those pairs satisfying $x^2 + y^2 < \Omega$. In such a case, Axiom 5 is violated. For example, let $\Omega = 1$, $x = y = 0.2$, and $z = 0.9$. Then $x^2 + y^2 = 0.08$, so (x, y) is in R_Ω; and $(x + y)^2 + z^2 = 0.97$, so $(x + y, z)$ is again in R_Ω; but $x^2 + (y + z)^2 = 1.25$, so $(x, y + z)$ is not in R_Ω, violating part of the conclusion to Axiom 5. This example shows clearly that Axiom 5 has a strong structural significance.

The other structural axioms are 2 and 7, whereas 1, 3, 4, 6, 8, and part of 5 $[(a \circ b) \circ c = a \circ (b \circ c)]$ are deducible from the representation (Theorem 4 below).

We remark also that the axiom system used here has two solvability conditions. One is regularity (Axiom 7), which requires solvability of the inequality $a \gtrsim b \circ c$, for c, given $a > b$. The other solvability condition is Axiom 2, which requires solvability of the equation, $c \circ d = e$, for e, whenever $a \gtrsim c$, $b \gtrsim d$, and $(a, b) \in B$. [The assertion $(c, d) \in B$ is equivalent to the assertion that the required e exists.]

We state two versions of the representation and uniqueness theorem. Theorem 4 gives the essential existence and uniqueness statements. A slight modification, Theorem 4′, asserts that any structure $\langle A, \gtrsim, B, \circ \rangle$ satisfying Axioms 1–8 can be mapped isomorphically into a structure $\langle Re^+, \geqslant, R_\Omega, + \rangle$, where R_Ω is defined as above.

THEOREM 4. *Let $\langle A, \gtrsim, B, \circ \rangle$ be an Archimedean, regular, positive, ordered local semigroup (Axioms 1–8 of Definition 2 all hold). Then there is a function ϕ from A to Re^+ such that for all $a, b \in A$,*

(i) *$a \gtrsim b$ iff $\phi(a) \geqslant \phi(b)$;*

(ii) *if $(a, b) \in B$, then $\phi(a \circ b) = \phi(a) + \phi(b)$.*

Moreover, if ϕ and ϕ' are any two functions from A to Re^+ satisfying conditions (i) *and* (ii), *then there exists $\alpha > 0$ such that for any nonmaximal $a \in A$,*

$$\phi'(a) = \alpha\phi(a).$$

The isomorphism version runs as follows:

THEOREM 4′. *Let the hypotheses of Theorem 4 hold and let ϕ be a function from A to Re^+ satisfying* (i) *and* (ii) *of that theorem. Let Ω be the least upper* *bound of* $\{\phi(a) \mid a \in A\}$. *Let* A' *be the set of nonmaximal elements of* A *and let* $B' = \{(a, b) \in B \mid a \circ b \in A'\}$. *Then* ϕ *is an isomorphism of* $\langle A', \succsim, B', \circ \rangle$ *into* $\langle \mathrm{Re}^+, \geqslant, R_\Omega, + \rangle$.

The point of Theorem 4′ is that ϕ not only carries $\succsim$ into $\geqslant$ and $\circ$ into $+$, but also carries B' into R_Ω, in the sense that

$$(a, b) \in B' \quad \text{iff} \quad (\phi(a), \phi(b)) \in R_\Omega .$$

If there is no maximal element in A, then $A' = A$ and $B' = B$. If there is a maximal element of form $a \circ b$, then the uniqueness statement of Theorem 4 and the isomorphism statement in Theorem 4′ can easily be extended to it. The only case where the restriction to nonmaximal elements matters is when the maximal element is not of the form $a \circ b$ for any $(a, b) \in B$.

2.2.2 Proof of Theorem 4 (Outline)

Because of the importance of this theorem and because the construction of the function ϕ embodies the basic processes of additive measurement, we first give a detailed sketch of the proof and then a full proof.

To approximate $\phi(b)/\phi(c)$, we take a small a and see how many copies of a are required to approximate b and how many to approximate c. If $ma \approx b$ and $na \approx c$ ($\approx$ means approximates), then $m/n \approx \phi(b)/\phi(c)$. This idea is justified by the desired properties of ϕ [(i) and (ii) of the theorem], since $ma \approx b$ should imply $m\phi(a) \approx \phi(b)$. Similarly, $n\phi(a) \approx \phi(c)$, so $m/n \approx \phi(b)/\phi(c)$. Thus, the first step of the proof is to define, for any a, b such that $b \succsim a$, an integer $N(a, b)$ such that $N(a, b)\, a \approx b$. Specifically, $N(a, b)$ is the largest integer such that $[N(a, b) - 1]a$ is defined and $[N(a, b) - 1]\, a \prec b$. The Archimedean property guarantees that such a largest integer exists.

Two cases must now be distinguished. If there is a smallest element a in A, then for every b, $N(a, b)\, a = b$. For if this were not true, we could use regularity (Axiom 7) to construct a' such that $b \succsim [N(a, b) - 1]\, a \circ a'$, and it would follow that $a' \prec a$, contrary to minimality of a. In this case, $\phi(b) = N(a, b)$ gives the required isomorphism into $\langle \mathrm{Re}^+, \geqslant, R_\Omega, + \rangle$.

In the second case, there is no least element of A, and the proof consists of showing that as a is taken smaller and smaller, $N(a, b)/N(a, c)$ converges, for every b, c, to a limit in Re^+. This limit is defined to be $\phi(b)/\phi(c)$. The limit exists because, for a' much smaller than a, $N(a', b) \approx N(a', a)\, N(a, b)$. This is intuitively obvious. If $N(a', a)$ copies of a' approximate a, and $N(a, b)$ copies of a approximate b, then $N(a', a)\, N(a, b)$ copies of a' approximate b. Thus, $N(a', b)/N(a', c) \approx N(a, b)/N(a, c)$, since the common factor $N(a', a)$ drops out when the approximation of b is divided by that of c. For example,

if the gradation is changed from feet (a) to millimeters (a'), then all approximate measurements will be multiplied approximately by $N(a', a)$, the number of millimeters per foot, and ratios remain approximately the same.

Another important feature of the proof is the method for taking a' sufficiently smaller than a. The trick is simple: take any $a' \prec a$, and then take a'' with $a \succsim a' \circ a''$ (regularity). The smaller of a' and a'' is then less than "half" of a.

Once ϕ is constructed, the rest of the proof [that properties (i) and (ii) and uniqueness hold] is easy. Additivity of ϕ, for example, follows using Axioms 3 and 4 to show that if $N(a, b)\, a \approx b$ and $N(a, c)\, a \approx c$, then $[N(a, b) + N(a, c)]\, a \approx b \circ c$. Ordering follows, using additivity. For if $b \succ c$, then for some a, $b \succsim c \circ a$. By construction of ϕ, $\phi(b) \geqslant \phi(c) + \phi(a)$, where $\phi(a) > 0$; thus, $\phi(b) > \phi(c)$.

Note the role played by each axiom of Definition 2. Axioms 2 and 5 characterize the local semigroup: the operation $\circ$ is defined and associative for all sufficiently small a, b. We can thus generate na from a, being unconcerned about the way that copies of a are associated in forming na. Axiom 6 guarantees that the standard sequence $a, 2a, \ldots$ increases steadily, while Axiom 8 guarantees that eventually, such a sequence will approximate any b. The "increasing" and "approximate" notions are based on the ordering (Axiom 1). Regularity (Axiom 7) plays several roles, as was pointed out in the above sketch. Finally, Axioms 3 and 4 play an important role in additivity.

Note that the axioms do not explicitly assume commutativity, $a \circ b = b \circ a$. Since this property is implied by Theorem 4, it must follow from the axioms, but it is not used in the course of the proof.

2.2.3 Preliminary Lemmas

In this section we collect a group of simple results concerning ordered local semigroups. The hypothesis common to all the lemmas is that $\langle A, \succsim, B, \circ \rangle$ is an ordered local semigroup (Axioms 1–5 of Definition 2). Also, N_a and na are defined as in Definition 2.

LEMMA 2. $m, n \in N_a$ *and* $(ma, na) \in B$ *iff* $m + n \in N_a$; *and when both conditions hold* $(ma) \circ (na) = (m + n)a$.

PROOF. Exercise 3. ◇

LEMMA 3. *If* $a \succsim a'$, $b \succsim b'$, *and* $(a, b) \in B$, *then* $a \circ b \succsim a' \circ b'$.

PROOF. Axioms 1–4. ◇

LEMMA 4. *If $a \succsim b$ and $n \in N_a$, then $n \in N_b$ and $na \succsim nb$.*

PROOF. From Lemma 3, by induction. ◇

LEMMA 5. *Let Axiom 6 hold. If $m, n \in N_a$, then $ma \succsim na$ iff $m \geqslant n$.*

PROOF. Obvious. ◇

LEMMA 6. *If $m \in N_a$, then $n \in N_{ma}$ iff $nm \in N_a$; in which case $n(ma) = (nm)a$.*

PROOF. The result is trivial for $n = 1$, $m \in I^+$. Suppose it holds for some $n \geqslant 1$, for all $m \in I^+$. If $n + 1 \in N_{ma}$, then

$$\begin{aligned}(n+1)(ma) &= [n(ma)] \circ (ma) \qquad \text{(Definition 2)}\\ &= [(nm)a] \circ (ma) \qquad \text{(inductive hypothesis).}\end{aligned}$$

By Lemma 2, $nm + m = (n+1)\,m \in N_a$, and $[(nm)\,a] \circ (ma) = [(n+1)\,m]a$. Therefore, if $n + 1 \in N_{ma}$, then $(n+1)\,m \in N_a$ and $(n+1)(ma) = [(n+1)\,m]a$.

Conversely, if $(n+1)\,m \in N_a$, then by applying Lemma 2 we have $((nm)a, ma) \in B$. By the inductive hypothesis, $n \in N_{ma}$ and $(nm)\,a = n(ma)$. Hence, $(n(ma), ma) \in B$. It follows from Definition 2 that $n + 1 \in N_{ma}$, as required.

By induction, the result holds for all n. ◇

2.2.4 Proof of Theorems 4 and 4′ (Details)

If $b \succ a$, then by Axiom 8, there exists a largest positive integer, denoted $N(a, b)$, such that $N(a, b) - 1 \in N_a$ and $b \succ (N(a, b) - 1)a$. We can let $N(a, a) = 1$ by definition.

First, suppose that there exists a minimal element $a \in A$. We show that for every $b \in A$, $N(a, b) \in N_a$ and $N(a, b)\,a = b$. By Axiom 7, for any $b \succ a$ there exists c such that $((N(a, b) - 1)\,a, c) \in B$ and $b \succsim [(N(a, b) - 1)\,a] \circ c$. By minimality of a, $c \succsim a$. By Axioms 1–3, we know that $N(a, b) \in N_a$ and $b \succsim N(a, b)a$. But then by maximality of $N(a, b)$, we cannot have $b \succ N(a, b)a$, hence, $b = N(a, b)a$. The same formula follows trivially if $b = a$.

Let $\phi(b) = N(a, b)$. We show that ϕ satisfies (i) and (ii) of Theorem 4. To prove (i), note that by Lemma 5, $N(a, b)a \succsim N(a, c)a$ iff $N(a, b) \geqslant N(a, c)$. Hence, $b \succsim c$ iff $\phi(b) \geqslant \phi(c)$. For (ii), note that

$$\begin{aligned}N(a, b \circ c)\,a &= b \circ c\\ &= [N(a, b)\,a] \circ [N(a, c)\,a]\\ &= [N(a, b) + N(a, c)]a \qquad \text{(Lemma 2).}\end{aligned}$$

By Lemma 5, $N(a, b \circ c) = N(a, b) + N(a, c)$, or $\phi(b \circ c) = \phi(b) + \phi(c)$. Finally, if ϕ' is any other function satisfying (i) and (ii), then for any b in A,

$$\begin{aligned}\phi'(b) &= \phi'(N(a, b)\,a)\\ &= N(a, b)\,\phi'(a) \qquad \text{[property (ii)]}\\ &= \alpha\phi(b),\end{aligned}$$

where $\alpha = \phi'(a) > 0$. This completes the proof of Theorem 4 for the case where a minimal element exists.

Next, suppose that A has no minimal element. We shall construct a sequence $a_1, \ldots, a_m, \ldots$ in A which converges to zero in the sense that for every $b \in A$, $N(a_m, b)$ is defined for sufficiently large m and diverges to $+\infty$. We then show that for any such sequence, $\lim_{m\to\infty} N(a_m, b)/N(a_m, c)$ exists in Re^+ for every $b, c \in A$. The limit is obviously independent of the sequence a_m, since any two sequences can be interleaved to form a third sequence with the same properties.

Let a_1 be arbitrary, and define the sequence inductively as follows. If a_m has been defined, choose $a_m' \prec a_m$. By Axiom 7, choose a_m'' such that $(a_m', a_m'') \in B$ and $a_m \succsim a_m' \circ a_m''$. Define $a_{m+1} = \min\{a_m', a_m''\}$ (the minimum is taken with respect to the ordering $\succsim$). By Lemma 3, for $m \geqslant 1$,

$$a_m \succsim a_m' \circ a_m'' \succsim 2a_{m+1}.$$

Since $a_{m+1} \succsim 2a_{m+2}$, we can apply Lemma 3 again to obtain

$$a_m \succsim 2a_{m+1} \succsim (2a_{m+2}) \circ (2a_{m+2}).$$

By Lemma 2, $a_m \succsim 4a_{m+2}$. Continuing this argument, we have by induction

$$a_m \succsim 2^n a_{m+n}, \qquad m \geqslant 1, \quad n \geqslant 0.$$

It goes without saying that the inductive argument, as developed above, includes the proof that $2^n \in N_{a_{m+n}}$, $m \geqslant 1$, $n \geqslant 0$.

Having defined the sequence a_m and established the basic property that $a_m \succsim 2^n a_{m+n}$, we now show that $N(a_m, b)$ is defined for sufficiently large m and approaches $+\infty$. Suppose first that $a_2 \succ b$. By definition, $a_2 \succ [N(b, a_2) - 1]b$. Since $(a_2, a_2) \in B$, by Axiom 2, $((N(b, a_2) - 1)\,b, b) \in B$, so $N(b, a_2) \in N_b$. By definition, $N(b, a_2)\,b \succsim a_2$. Now choose m so large that $2^m > N(b, a_2)$. We show that the supposition that $a_{m+2} \succsim b$ leads to a contradiction. In fact, if $a_{m+2} \succsim b$, then by Lemma 4, $2^m \in N_b$ and we have

$$N(b, a_2)\,b \succsim a_2 \succsim 2^m a_{m+2} \succsim 2^m b.$$

But by Lemma 5, $2^m > N(b, a_2)$ implies $2^m b \succ N(b, a_2)b$, a contradiction.

Thus, we have $b \succ a_{m+2}$. We established this formula assuming $a_2 \succ b$; but otherwise, the same formula holds with $m = 1$. Thus, for every b, there exists $m \geqslant 1$ such that for all $n \geqslant 0$, $b \succ a_{m+n+2}$. We now have

$$\begin{aligned} b &\succ [N(a_{m+2}, b) - 1]\, a_{m+2} && \text{(definition)}\\ &\succsim [N(a_{m+2}, b) - 1](2^n a_{m+n+2}) && \text{(Lemma 4)}\\ &= (2^n[N(a_{m+2}, b) - 1])\, a_{m+n+2} && \text{(Lemma 6).} \end{aligned}$$

By definition, therefore, $N(a_{m+n+2}, b) > 2^n[N(a_{m+2}, b) - 1]$. It follows that $N(a_{m+n+2}, b) \to +\infty$ as $n \to \infty$, as required.

To evaluate $N(a_m, b)/N(a_m, c)$, as $m \to \infty$, we need the following inequality:

If $(a, a') \in B$ *and* $b \succsim a \succsim a'$, *then*

$$N(a', a)\, N(a, b) > N(a', b) - 1 \geqslant [N(a', a) - 1][N(a, b) - 1].$$

For simplicity in proving this inequality, we denote $N(a', a)$ by $m + 1$ and $N(a, b)$ by $n + 1$. Note that if $b = a$ or $a = a'$ (or both), the inequality is trivial; so we can assume $b \succ a \succ a'$, hence, $m, n \geqslant 1$. We have $a \succ ma'$ and $(a, a') \in B$, so $m + 1 \in N_{a'}$ and $(m + 1)\, a' \succsim a$. Suppose that $(n + 1)(m + 1) \in N_{a'}$. By Lemma 6, $n + 1 \in N_{(m+1)a'}$; thus by Lemma 4, $n + 1 \in N_a$. By definition, $(n + 1)\, a \succsim b$, hence $(n + 1)(m + 1)\, a' \succsim b$. This shows that either $(n + 1)(m + 1) \notin N_{a'}$, or else $(n + 1)(m + 1)\, a' \succsim b$; in either case, we have $(n + 1)(m + 1) \geqslant N(a', b)$, which gives the left half of the inequality. From $b \succ na$ and $a \succ ma'$ we have $nm \in N_{a'}$ and $b \succ nma'$ (Lemmas 4 and 6). Thus, $N(a', b) - 1 \geqslant nm$, which is the right half of the inequality.

From this inequality, it follows that for any b, c, for all sufficiently large values of m, and for all $n \geqslant m$,

$$\frac{[N(a_n, b) - 1]}{[N(a_n, c) - 1]} < \frac{N(a_n, a_m)\, N(a_m, b)}{[N(a_n, a_m) - 1][N(a_m, c) - 1]}.$$

(Use the upper bound for the numerator and the lower bound for the denominator.) The first consequence of this is that for all $n \geqslant m$,

$$\frac{[N(a_n, b) - 1]}{[N(a_n, c) - 1]} \leqslant 2\, \frac{N(a_m, b)}{[N(a_m, c) - 1]}.$$

Thus, the sequence $[N(a_n, b) - 1]/[N(a_n, c) - 1]$ is bounded above for arbitrary b, c. By interchanging the roles of b and c, we prove that the sequence of inverses has a finite upper bound and, hence, the sequence itself has a positive lower bound. Let L^* and L_* be, respectively, the greatest and least limit points of the sequence; then $0 < L_* \leqslant L^* < \infty$. Holding m fixed and letting $n \to \infty$ in our inequality, and noting that $N(a_n, a_m)/[N(a_n, a_m) - 1]$ converges to 1, we obtain

$$L^* \leqslant N(a_m, b)/[N(a_m, c) - 1].$$

Now letting $m \to \infty$ in this last inequality, we see that $L^* \leqslant L_*$, i.e., $L^* = L_*$ and the sequence $[N(a_n, b) - 1]/[N(a_n, c) - 1]$, hence also the sequence $N(a_n, b)/N(a_n, c)$ converges to a limit in Re^+.

For any $b \in A$, define

$$\phi(b) = \lim_{m \to \infty} N(a_m, b)/N(a_m, a_1).$$

Since $\phi(a_1) = 1$, a_1 has been taken as the unit of measurement; this choice is arbitrary.

We show that ϕ is additive (ii) and order preserving (i). Suppose that $(b, c) \in B$. For m such that $b, c \succ a_m$, we have $b \succ [N(a_m, b) - 1]\, a_m$, $c \succ [N(a_m, c) - 1]\, a_m$, hence, $b \circ c \succsim [N(a_m, b) + N(a_m, c) - 2]\, a_m$ (Lemmas 2 and 3). If $N(a_m, b) + N(a_m, c) \in N_{a_m}$, then clearly

$$[N(a_m, b) + N(a_m, c)]\, a_m \succsim b \circ c.$$

Thus we have the inequality

$$N(a_m, b) + N(a_m, c) \geqslant N(a_m, b \circ c) > N(a_m, b) + N(a_m, c) - 2.$$

Dividing through by $N(a_m, a_1)$ and letting $m \to \infty$ yields $\phi(b) + \phi(c) = \phi(b \circ c)$. Thus, (ii) holds.

Suppose that $b \succ c$. By Axiom 7, there exists c' with (c, c') in B and $b \succsim c \circ c'$. For each m, $N(a_m, b) \geqslant N(a_m, c \circ c')$; hence,

$$\begin{aligned} \phi(b) &\geqslant \phi(c \circ c')\\ &= \phi(c) + \phi(c') && \text{[by (ii)]}\\ &> \phi(c) && [\phi(c') > 0]. \end{aligned}$$

It follows from this that $b \succsim c$ if and only if $\phi(b) \geqslant \phi(c)$, as required for (i).

Finally, we establish uniqueness. Suppose that ϕ' is any other function from A to Re^+ satisfying (i) and (ii). If b is nonmaximal in A, then by Axiom 7, there exists c with $(b, c) \in B$. For $a_m \precsim c$, $N(a_m, b) \in N_{a_m}$ and $N(a_m, b)\, a_m \succsim b$. By (i) and (ii),

$$N(a_m, b)\, \phi'(a_m) \geqslant \phi'(b) > [N(a_m, b) - 1]\, \phi'(a_m).$$

Since a_2 is nonmaximal, we have for $m > 2$

$$N(a_m, a_2)\, \phi'(a_m) \geqslant \phi'(a_2) > [N(a_m, a_2) - 1]\, \phi'(a_m).$$

Dividing these inequalities yields

$$N(a_m, b)/[N(a_m, a_2) - 1] > \phi'(b)/\phi'(a_2) > [N(a_m, b) - 1]/N(a_m, a_2).$$

Letting $m \to \infty$ yields $\phi'(b)/\phi'(a_2) = \phi(b)/\phi(a_2)$, or $\phi'(b) = \alpha\phi(b)$, where $\alpha = \phi'(a_2)/\phi(a_2) > 0$.

Note that the uniqueness statement can be extended to a maximal element $a \in A$ (if any) only if $a = b \circ c$ for some b, c. This is necessarily the case if there is a minimal element in A, but need not hold otherwise.

To prove Theorem 4′, let Ω and R_Ω be defined as in that theorem. To show that ϕ is an isomorphism, it suffices to establish that

$$(a, b) \in B' \quad \text{iff} \quad (\phi(a), \phi(b)) \in R_\Omega .$$

One direction is obvious: if $(a, b) \in B'$, then $a \circ b \in A'$, and so $\phi(a) + \phi(b) < \Omega$. Conversely, suppose that $\phi(a) + \phi(b) < \Omega$. Since Ω is a least upper bound, there exists $c \in A$ such that $\phi(a) + \phi(b) < \phi(c)$. If there is a minimal element $a_1 \in A$, then $N(a_1, a) + N(a_1, b) \in N_{a_1}$, hence, by an application of Lemma 2, $(N(a_1, a)\, a_1, N(a_1, b)\, a_1) \in B'$, or $(a, b) \in B'$. If there is no minimal element in A, we know that $\phi(a_m) \to 0$, where a_m is the sequence constructed in the proof of Theorem 4. Choose a_m with $\phi(a_m) < \frac{1}{2}[\phi(c) - \phi(a) - \phi(b)]$. It follows readily that

$$N(a_m, a) + N(a_m, b) \in N_{a_m},$$

and hence, that $(N(a_m, a)\, a_m, N(a_m, b)\, a_m) \in B$. Thus, $(a, b) \in B'$. ◇

From the proof of the uniqueness theorem, we have a good idea of the precision of approximations to $\phi(b)$. In fact, the inequality used to prove uniqueness can be rewritten as follows.

$$\frac{N(a_m, b)}{N(a_m, a_2)} \cdot \frac{N(a_m, a_2)}{N(a_m, a_2) - 1} > \frac{\phi(b)}{\phi(a_2)} > \frac{N(a_m, b) - 1}{N(a_m, b)} \cdot \frac{N(a_m, b)}{N(a_m, a_2)}.$$

Hence, the proportion of error in estimating $\phi(b)/\phi(a_2)$ by $N(a_m, b)/N(a_m, a_2)$ is not greater than the larger of $1/N(a_m, a_2)$ and $1/N(a_m, b)$. These error limits are known, and approach zero at least as fast as 2^{-m} (by construction of the sequence a_m). Hence, we can specify precisely the finite observations required to obtain any preassigned accuracy. In this way, our proof does not have the undesirable nonconstructive features of the definition of ϕ in the proof of Theorem 2 (Section 2.1).

2.2.5 Archimedean Ordered Groups

As a corollary to Theorem 4, we obtain an isomorphism for the case of groups.

DEFINITION 3. *Let $\langle A, \succsim\rangle$ be a simple order and $\circ$ a binary operation on A such that $\langle A, \circ\rangle$ is a group. The triple $\langle A, \succsim, \circ\rangle$ is called a* simply ordered group *provided that for all $a, b, c \in A$, if $a \succsim b$, then $a \circ c \succsim b \circ c$ and $c \circ a \succsim c \circ b$. Let the group identity be denoted e. The group is* Archimedean *provided that if $a \succ e$ and $b \in A$, then $na \not\succ b$ for some $n \in I^+$.*

THEOREM 5. *Let $\langle A, \succsim, \circ\rangle$ be an Archimedean simply ordered group. Then $\langle A, \succsim, \circ\rangle$ is isomorphic to a subgroup of $\langle \mathrm{Re}, \geqslant, +\rangle$, and if ϕ, ϕ' are any two isomorphisms, $\phi' = \alpha\phi$ for some $\alpha > 0$.*

PROOF. Let $A^+ = \{a \mid a \succ e\}$, where e is the identity. Let $B = A^+ \times A^+$, and let $\succsim^+$ be the restriction of $\succsim$ to A^+. If $a, b \in A^+$, then $a \circ b \succsim a \circ e = a \succ e$, so $a \circ b \in A^+$. Thus, $\circ$ induces a function $\circ^+$ from B into A^+. We show that $\langle A^+, \succsim^+, B, \circ^+\rangle$ satisfies Axioms 1–8 of Definition 2. In fact, Axioms 1–5 and 8 are immediate. To show positivity (Axiom 6), suppose that $c \in A^+$ and $a \succsim a \circ c$. Then

$$e = a^{-1} \circ a \succsim a^{-1} \circ (a \circ c) = c,$$

contradicting $c \in A^+$. Thus, $a \circ c \succ a$, as required. For regularity (Axiom 7), suppose that $c \succ a$ and let $b = c \circ a^{-1}$, where a^{-1} is the inverse of a. If $e \succsim b$, then

$$a = e \circ a \succsim b \circ a = c,$$

a contradiction, so $b \in A^+$; and $c = b \circ a$.

We let ϕ^+ be a function from A^+ to Re^+ satisfying (i) and (ii) of Theorem 4, and extend it to A by noting that if $e \succ a$, then $a^{-1} \in A^+$ (by the same argument as for regularity, above). Therefore, define ϕ by

$$\phi(a) = \begin{cases} \phi^+(a), & a \in A^+, \\ 0, & a = e, \\ -\phi^+(a^{-1}), & e \succ a. \end{cases}$$

The remainder of the theorem is easy to verify. ◇

2.2.6 Note on Hölder's Theorem

Hölder (1901) presented a set of seven axioms that are necessary and sufficient for an isomorphism *onto* $\langle \mathrm{Re}^+, \geqslant, +\rangle$. Thus, he also dealt with the case of an ordered semigroup. He used much stronger structural assumptions, including (in our notation) $B = A \times A$, no minimal element, solvability of $c = a \circ b$ for a given $c \succ b$ and for b given $c \succ a$, and the Dedekind property (if $A = C \cup D$, where C, D are nonempty, $C \cap D = \varnothing$, and $c \in C$, $d \in D$ imply $c \prec d$, then either sup C or inf D exists). These are, of course, all necessary for an isomorphism *onto* Re$^+$. With these structural properties and a slightly stronger positivity assumption, he did not need our Axioms 2–4 and 8. Since the axioms follow easily from his assumptions, it is straightforward to prove his result as a corollary to Theorem 4.

The proofs given by Birkhoff (1967, p. 300) and Fuchs (1963, p. 45) of Theorem 5, which is now often called Hölder's theorem, are essentially the same as Hölder's proof. They all suffer from the disadvantage that the method of constructing ϕ could not be used in actual measurement, because it requires that na be defined for every n. This means that the proof does not generalize to the case of more realistic structural assumptions.

2.2.7 Archimedean Ordered Semirings

Suppose that $\langle A, \succsim \rangle$ is a simple order, B, B^* are two subsets of $A \times A$, and $\oplus, *$ are two binary operations from B to A and from B^* to A, respectively, such that $\langle A, \succsim, B, \oplus \rangle$ and $\langle A, \succsim, B^*, * \rangle$ are both ordered local semigroups. If they are both Archimedean, regular, and positive, then two, possibly unrelated, isomorphisms ϕ and ϕ^* can be constructed into Re$^+$. If the two operations are linked, however, by the distributive laws of multiplication over addition, i.e.,

$$(a \oplus b) * c = (a * c) \oplus (b * c),$$
$$c * (a \oplus b) = (c * a) \oplus (c * b),$$

then it is desirable to construct a single isomorphism ϕ that is both additive and multiplicative. That is, in addition to satisfying (i) and (ii) of Theorem 4, ϕ must satisfy $\phi(a * b) = \phi(a)\,\phi(b)$, and thus yield an isomorphism of $\langle A, \succsim, B, B^*, \oplus, * \rangle$ into $\langle \mathrm{Re}^+, \geqslant, R_\Omega, R_\Omega^*, +, \cdot \rangle$, where $+, \cdot$ are ordinary addition and multiplication in Re$^+$.

The requirement that ϕ be order preserving and additive already determines it up to ratio-scale transformations (Theorem 4). If A has a multiplicative identity e, then from $a = a * e$ we have

$$\phi(a) = \phi(a * e) = \phi(a)\,\phi(e),$$

whence $\phi(e) = 1$. Thus, the additional requirement that ϕ be multiplicative determines the unit of measurement. We show below that even if A has no multiplicative identity, there exists nevertheless one and only one additive representation that is also multiplicative. In most applications of this theorem, however, one still obtains ratio-scale, rather than absolute, measurement; the reason becomes clear in Chapter 7.

DEFINITION 4. *Suppose that A is a set, $\succsim$, B, and B^* are three binary relations on A, $\oplus$ is a function from B into A, and $*$ is a function from B^* into A. The sextuple $\langle A, \succsim, B, B^*, \oplus, * \rangle$ is an* ordered local semiring *iff the following four axioms are satisfied:*

1. *$\langle A, \succsim, B, \oplus \rangle$ is an ordered local semigroup (Definition 2, Axioms 1–5).*
2. *$\langle A, \succsim, B^*, * \rangle$ satisfies Axioms 1–4 of Definition 2 and the following modified version of Axiom 5:*
   5′. *If $(a, b), (b, c) \in B^*$, then $(a * b, c) \in B^*$ iff $(a, b * c) \in B^*$; and if both conditions hold, then $(a * b) * c = a * (b * c)$.*
3. (i) *If $(b, c) \in B$ and $(a, b \oplus c) \in B^*$, then $(a, b), (a, c) \in B^*$, $(a * b, a * c) \in B$, and $a * (b \oplus c) = (a * b) \oplus (a * c)$.*
   (ii) *If $(a, b) \in B$ and $(a \oplus b, c) \in B^*$, then $(a, c), (b, c) \in B^*$, $(a * c, b * c) \in B$, and $(a \oplus b) * c = (a * c) \oplus (b * c)$.*
4. *For any $a \in A$, there exists $(b, c) \in B$ such that $(a, b \oplus c) \in B^*$.*

The local semigroup mentioned in Axiom 1 is termed the additive ($\oplus$) semigroup; the semiring is defined to be positive, regular, *or* Archimedean *according to whether these properties are satisfied by the additive semigroup (see Definition 2, Axioms 6–8).*

The numerical ordered local semigroup described in Section 2.2.1 can be made into an ordered local semiring by putting

$$R_\Omega^* = \{(x, y) \mid x, y \in \mathrm{Re}^+ \quad \text{and} \quad x \cdot y < \Omega\}.$$

It is easy to verify that $\langle \mathrm{Re}^+, \geqslant, R_\Omega, R_\Omega^*, +, \cdot \rangle$ satisfies Axioms 1–4 of Definition 4. Note that the replacement of Axiom 5 of Definition 2 by 5′, in Axiom 2 of this definition, is necessary, since $x \cdot y < \Omega$ and $x \cdot y \cdot z < \Omega$ do not entail that $y \cdot z < \Omega$.

The representation and uniqueness theorem is given by the following:

THEOREM 6. *Let $\langle A, \succsim, B, B^*, \oplus, * \rangle$ be an Archimedean, regular, positive, ordered local semiring (see axioms of Definitions 4 and 2). Then there is a unique function ϕ from A to Re$^+$ such that, for all $a, b \in A$:*

(i) *$a \succsim b$ iff $\phi(a) \geqslant \phi(b)$;*
(ii) *if $(a, b) \in B$, then $\phi(a \oplus b) = \phi(a) + \phi(b)$;*
(iii) *if $(a, b) \in B^*$, then $\phi(a * b) = \phi(a)\,\phi(b)$.*

PROOF. The proof is based on the method followed by Fuchs (1963, p. 126) to treat Archimedean ordered rings. A less abstract proof is sketched in Exercises 4–13. The actual construction of an isomorphism is probably better based on those exercises, although it could be carried out using the proof given here.

For any $a \in A$, let $A_a = \{b \mid (a, b) \in B^*\}$ and $B_a = \{(b, c) \mid (b, c) \in B$ and $b \oplus c \in A_a\}$. Let $\oplus_a$ and $\succsim_a$ be $\oplus$ and $\succsim$ restricted to B_a and to A_a, respectively. Then it is easy to show that $\langle A_a, \succsim_a, B_a, \oplus_a\rangle$ is an Archimedean, regular, positive, ordered local semigroup.

First, note that Axiom 4 is precisely the statement that A_a and B_a are nonempty for any $a \in A$. Obviously, $\oplus_a$ is a function from B_a to A_a.

Axioms 1, 3, 4, 6, and 8 of Definition 2 are immediate, since they hold for $\langle A, \succsim, B, \oplus\rangle$ and are true *a forteriori* for any subsets.

If $(b, c) \in B_a$, $b \succsim b'$, and $c \succsim c'$, then $b \oplus c \succsim b' \oplus c'$. Hence, by Axiom 2 of Definition 2, applied to $\langle A, \succsim, B^*, *\rangle$, $(a, b' \oplus c') \in B^*$, i.e., $(b', c') \in B_a$. This establishes Axiom 2 of Definition 2 for $\langle A_a, \succsim_a, B_a, \oplus_a\rangle$.

Axiom 5 of Definition 2 follows readily from the same axiom applied to $\langle A, \succsim, B, \oplus\rangle$: If (b, b') and $(b \oplus b', b'') \in B_a$, then we have (b', b'') and $(b, b' \oplus b'') \in B$, and $(b \oplus b') \oplus b'' = b \oplus (b' \oplus b'')$. Thus,

$$(a, b \oplus (b' \oplus b'')) \in B^*,$$

so $(b, b' \oplus b'') \in B_a$. By Axiom 6, $b \oplus (b' \oplus b'') \succ b' \oplus b''$, so

$$(a, b' \oplus b'') \in B^*,$$

hence $(b', b'') \in B_a$.

Finally, we show that Axiom 7 of Definition 2 holds. Suppose that $b, c \in A_a$ and $b \succ c$. We can find $c' \in A$ such that $b \succsim c \oplus c'$. Since $(a, c \oplus c') \in B^*$, $(c, c') \in B_a$, as required.

Note that, so far, Axiom 3 of Definition 4 has not been used.

By Theorem 4, there exists a function ψ from A to Re^+ satisfying (i) and (ii) for $\langle A, \succsim, B, \oplus\rangle$. *A forteriori*, ψ satisfies (i) and (ii) for each $\langle A_a, \succsim_a, B_a, \oplus_a\rangle$. However, we can define a new representation for the latter semigroup by

$$\psi_a(b) = \psi(a * b).$$

To prove that ψ_a satisfies (ii), take $(b, c) \in B_a$. Then $(a, b \oplus c) \in B^*$, so by Axiom 3 of Definition 4, (a, b) and $(a, c) \in B^*$, $(a * b, a * c) \in B$, and $a * (b \oplus c) = (a * b) \oplus (a * c)$. Hence,

$$\begin{aligned}\psi_a(b \oplus c) &= \psi[a * (b \oplus c)]\\ &= \psi[(a * b) \oplus (a * c)]\\ &= \psi(a * b) + \psi(a * c)\\ &= \psi_a(b) + \psi_a(c).\end{aligned}$$

To prove that ψ_a satisfies (i), suppose that $b, c \in A_a$ and $b \succ c$. Choose $c' \in A_a$ such that $b \succsim c \oplus c'$. Then $a * b \succsim a * (c \oplus c')$. Hence,

$$\begin{aligned}\psi_a(b) &= \psi(a * b)\\ &\geqslant \psi[a * (c \oplus c')]\\ &= \psi_a(c \oplus c')\\ &= \psi_a(c) + \psi_a(c')\\ &> \psi_a(c),\end{aligned}$$

as required.

By the uniqueness part of Theorem 4, there is a positive constant, denoted $\phi(a)$, such that for all nonmaximal $b \in A_a$,

$$\psi_a(b) = \phi(a)\, \psi(b).$$

We now show that ϕ satisfies conditions (i)–(iii).

Suppose that $(a, a') \in B$. By Axiom 4 of Definition 4, we can choose a nonmaximal $b \in A_{a \oplus a'}$. Clearly, b is also a nonmaximal element of A_a and of $A_{a'}$. We have

$$\begin{aligned}\phi(a \oplus a')\, \psi(b) &= \psi_{a \oplus a'}(b)\\ &= \psi[(a \oplus a') * b]\\ &= \psi[(a * b) \oplus (a' * b)]\\ &= \psi(a * b) + \psi(a' * b)\\ &= \psi_a(b) + \psi_{a'}(b)\\ &= \phi(a)\, \psi(b) + \phi(a')\, \psi(b)\\ &= [\phi(a) + \phi(a')]\, \psi(b).\end{aligned}$$

Hence, ϕ satisfies (ii).

Suppose that $a \succ a'$; let $a \succsim a' \oplus a''$. Choose nonmaximal $b \in A_a$ and $A_{a' \oplus a''}$; then

$$\begin{aligned}\phi(a)\, \psi(b) &= \psi_a(b)\\ &= \psi(a * b)\\ &\geqslant \psi[(a' \oplus a'') * b]\\ &= \phi(a' \oplus a'')\, \psi(b)\\ &= [\phi(a') + \phi(a'')]\, \psi(b).\end{aligned}$$

Hence, $\phi(a) > \phi(a')$.

Finally, we prove (iii). Take $(a, a') \in B^*$. Choose nonmaximal $b' \in A_{a * a'}$ and nonmaximal $b'' \in A_a$. Let $b = \min\{b', b''\}$. Then b is nonmaximal in both $A_{a*a'}$ and $A_{a'}$, and clearly, $a' * b$ is nonmaximal in A_a. We therefore have

$$\begin{aligned}\phi(a * a')\,\psi(b) &= \psi_{a*a'}(b)\\ &= \psi[(a * a') * b]\\ &= \psi[a * (a' * b)]\\ &= \psi_a(a' * b)\\ &= \phi(a)\,\psi(a' * b)\\ &= \phi(a)\,\psi_{a'}(b)\\ &= \phi(a)\,\phi(a')\,\psi(b).\end{aligned}$$

Thus, $\phi(a * a') = \phi(a)\,\phi(a')$.

For uniqueness, note that if $\alpha\phi$ satisfies (i)–(iii) of Theorem 6, then by (iii), $\alpha\phi(a * b) = [\alpha\phi(a)] \cdot [\alpha\phi(b)]$, whence $\alpha = \alpha^2$ or $\alpha = 1$. This argument applies except possibly for maximal elements. But if a is maximal in A, we can choose b such that (a, b) is in B^* and such that b, $a * b$ are nonmaximal. We now have $\phi(a) = \phi(a * b)/\phi(b)$, giving uniqueness of ϕ at a. ◇

We next establish a corollary to Theorem 6 which is the classic result in this area (see Fuchs, 1963, p. 126); it stands to Theorem 6 as Hölder's Theorem 5 does to Theorem 4.

DEFINITION 5. *Suppose that A is a set, $\succsim$ a binary relation on A, and $\oplus$ and $*$ binary operations on A. Then $\langle A, \succsim, \oplus, *\rangle$ is an* Archimedean ordered ring *provided that:*

(i) *$\langle A, \oplus, *\rangle$ is a ring with zero element θ;*
(ii) *$\langle A, \succsim, \oplus\rangle$ is an Archimedean ordered group;*
(iii) *if $a \succ \theta$ and $b \succ c$, then $a * b \succ a * c$ and $b * a \succ c * a$.*

COROLLARY TO THEOREM 6. *An Archimedean ordered ring is uniquely isomorphic to a subring of $\langle \mathrm{Re}, \geqslant, +, \cdot\rangle$.*

PROOF. By Theorem 5, there is an isomorphism ϕ' of $\langle A, \succsim, \oplus\rangle$ into a subgroup of $\langle \mathrm{Re}, \geqslant, +\rangle$. Let $A^+ = \{a \mid a \succ \theta\}$. According to Exercise 14, the restriction of $\succsim$, $\oplus$, and $*$ to A^+ forms an Archimedean, regular, positive, ordered, local semiring (with $B = B^* = A^+ \times A^+$). Let ϕ be the unique isomorphism of Theorem 6. By the uniqueness assertion of Theorem 5, there exists $\alpha > 0$ such that over A^+, $\phi = \alpha\phi'$. Extend ϕ to all of A by defining $\phi = \alpha\phi'$. For all $a \prec \theta$, $\phi(a) = \alpha\phi'(a) = -\alpha\phi'(-a) = -\phi(-a)$. From this and the fact that ϕ preserves $*$ over A^+, it is trivial to show that it preserves $*$ over all of A. ◇

2.3 FINITE SETS OF HOMOGENEOUS LINEAR INEQUALITIES

The general problem of this section is to find solutions to families of inequalities and equations of the form

$$\begin{aligned}\sum_{j=1}^{n} \alpha_{ij}x_j &> 0, \qquad i = 1,\ldots, m',\\ \sum_{j=1}^{n} \beta_{ij}x_j &= 0, \qquad i = 1,\ldots, m''.\end{aligned} \tag{1}$$

Such a family of inequalities and equations arises in measurement contexts when the $x_1, \ldots, x_n$ are unknown values of quantities to be measured, and each inequality or equation of System (1) is entailed, via a linear measurement model, by a corresponding observation of order or equality between two objects (see Section 1.1.3 for examples). Any solution to System (1) provides a measurement scale compatible with the given observations of order and equality. In all applications of System (1) in this book, the coefficients α_{ij}, β_{ij} are integers (see Chapter 9).

2.3.1 Intuitive Explanation of the Solution Criterion

We shall analyze the System (1) using vector methods. The desired solution to System (1), $(x_1, \ldots, x_n)$, is a vector, and the coefficients of any inequality or equation of the system also form a vector, e.g., $(\alpha_{i1}, \ldots, \alpha_{in})$. For convenience, we abbreviate $(x_1, \ldots, x_n)$ by x, $(\alpha_{i1}, \ldots, \alpha_{in})$ by α_i, and $(\beta_{i1}, \ldots, \beta_{in})$ by β_i. Other abbreviations of the same sort are introduced later.

The *angle* between two vectors x and y is the angle whose vertex is the *origin* $\mathbf{0} = (0,\ldots, 0)$, or $(0, 0, 0)$ in three dimensions, and whose sides are the lines from $\mathbf{0}$ through each vector. The *scalar product* of two vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ is simply the sum of the products of their respective components, i.e., $\sum_{i=1}^{n} x_i y_i$. The reader can easily verify (in the two-dimensional case, for example) that the angle between x and y is acute, obtuse, or right depending on whether their scalar product is, respectively, positive, negative, or zero. In terms of these concepts, System (1) can be restated as

$$\begin{aligned}&x \quad \text{makes an acute angle with each } \alpha_i, \qquad i = 1,\ldots, m',\\ &x \quad \text{makes a right angle with each } \beta_i, \qquad i = 1,\ldots, m''.\end{aligned} \tag{1'}$$

Suppose that $n = 3$, that $m'' = 2$, and that β_1, β_2 are distinct from $\mathbf{0}$ and have a nonzero angle between them. The three points, $\mathbf{0}$, β_1, β_2 lie in a unique plane, and if x solves (1′), then the line from $\mathbf{0}$ through x must be at right angles to that plane. Thus, the direction of x from $\mathbf{0}$ is completely determined except for the choice of which side of the plane x is on. If we choose x on one side of the plane, then it makes an acute angle with any other point on the same side, a right angle with any point in the plane, and an obtuse angle with any point on the opposite side. Hence, it is obvious that in this case, *a solution to System* (1′) *can be found iff all the points* α_i *lie on the same side of the plane determined by* $\mathbf{0}, \beta_1, \beta_2$.

Let us keep $n = 3$ and consider values of m'' other than 2. If $m'' \geqslant 3$, there are two possibilities. If all the β_i lie in a single plane (determined by two of them with $\mathbf{0}$), then we take x on a line perpendicular to this plane at $\mathbf{0}$, and the situation is as before. Otherwise, they do not lie in a single plane, and so no x can be found which forms a right angle with all the β_i. Hence the only vector x which satisfies the equations is $\mathbf{0}$, and this does not satisfy the strict inequalities. Thus, System (1′) has no solution (unless there are no inequalities).

If $m'' = 1$, then $\mathbf{0}$ and β_1 can be part of an infinite number of different planes. If all the α_i lie on the same side of just one of these planes, then an x perpendicular to that plane solves System (1′).

The extreme case is $m'' = 0$ in which there are no equalities. This case is common in the practice of measurement—it means that no two objects were accepted as exactly equal. Here, it suffices that there be any plane through $\mathbf{0}$ such that all the α_i lie on the same side of it; there are no β_i to constrain the position of the plane.

The analysis is identical for $n \geqslant 3$, except that the equivalent of a plane—a hyperplane—is determined by $\mathbf{0}$ plus $n - 1$ independent β_i, if such exist.

And so we reduce the problem of solving System (1) or (1′) to finding a plane (or a hyperplane) through $\mathbf{0}$ and through all the β_i (if any), such that all the α_i lie on the same side of it. A solution x is just a vector such that the line through $\mathbf{0}$ and x is perpendicular to the plane and x is on the same side as the α_i.

To advance the problem further, we begin with the case $m'' = 0$. Is there a plane through $\mathbf{0}$ such that all α_i are on one side of it? If so, consider the points α_i as the vertices of a polyhedron. This polyhedron lies all on one side of the plane, and so, in particular, $\mathbf{0}$ is outside it. Conversely, if $\mathbf{0}$ is not in the polyhedron with the α_i as vertices, then we can pass a plane through $\mathbf{0}$ such that the whole polyhedron is on one side of it. It turns out that this is precisely the criterion for solvability of System (1′) if $m'' = 0$: $\mathbf{0}$ must be exterior to the polyhedron generated by the α_i. In fact, in this case, one way to obtain a solution is to choose the point on the polyhedron that is closest to $\mathbf{0}$. This point is a solution x. For through this point, we can pass a plane tangent to the polyhedron (in the sense that it does not enter it),

and the parallel plane through $\mathbf{0}$ has the polyhedron on one side of it. The line from $\mathbf{0}$ to x is perpendicular to both planes. This is illustrated, in a two-dimensional cross section, in Figure 1.

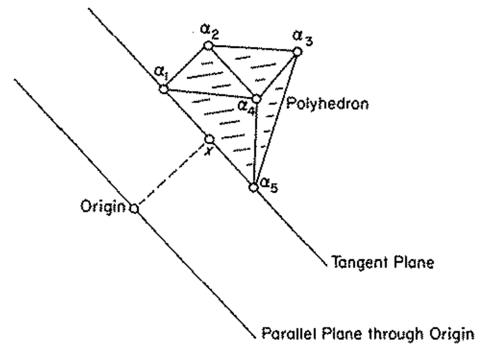

FIGURE 1. Polyhedron (shaded area) generated by five inequality vectors, $\alpha_1, \ldots, \alpha_5$, and the solution x with planes perpendicular to x (two-dimensional cross section).

The analysis for $m'' > 0$ is similar. If the plane through $\mathbf{0}$ is constrained to contain β_1 ($m'' = 1$), then we must require that the α-polyhedron not intersect the (infinite) line through $\mathbf{0}$ and β_1. If the plane through $\mathbf{0}$ is determined by β_1, β_2, then the α-polyhedron must not intersect this plane.

For $n > 3$, the analysis is analogous to the one just given. The only difference is that there are more possibilities for the number of independent β_i, other than 0, 1, or 2. If there are $n - 1$ independent equality vectors β_i, then they determine an $n - 1$ dimensional hyperplane, to which the solution x must be perpendicular. All α_i must lie on the same side of this hyperplane. In general, any number of independent β_i, not exceeding $n - 1$, generate a subspace, and this subspace partially constrains the position for a hyperplane with the polyhedron of inequality vectors lying to one side of it. The general theorem we shall prove, then, is the following:

THEOREM 7 (Intuitive version). *System* (1′) *has a solution iff the polyhedron generated by the vectors* α_i *does not intersect the subspace generated by the vectors* β_i (*when* $m'' = 0$, *the subspace in question is taken equal to* $\mathbf{0}$).

2.3.2 Vector Formulation and Preliminary Lemmas

We now use vector concepts to give a precise formulation of Theorem 7, and as a preliminary to its proof, we state and prove a fundamental lemma of linear programming theory. Our treatment closely follows Gale (1960). Readers whose background in linear algebra is insufficient for them to follow this section can easily substitute pp. 28–47 of Gale's book for this section.

The *subspace* generated by vectors $\beta_1, \ldots, \beta_r$, where $\beta_i = (\beta_{i1}, \ldots, \beta_{in})$, is the set of all vectors of form

$$\sum_{i=1}^{r} \lambda_i \beta_i = \left(\sum_{i=1}^{r} \lambda_i \beta_{i1}, \ldots, \sum_{i=1}^{r} \lambda_i \beta_{in}\right),$$

where $\lambda_1, \ldots, \lambda_r$ are arbitrary real numbers.

The *convex hull* of vectors $\alpha_1, \ldots, \alpha_s$, where $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{in})$, is the set of all vectors of form

$$\sum_{i=1}^{s} \lambda_i \alpha_i,$$

where $\lambda_1, \ldots, \lambda_s$ are restricted to be greater than or equal to 0 and $\sum_{i=1}^{s} \lambda_i = 1$. This concept replaces the intuitive concept of polyhedron used above. We denote the scalar product of vectors x, y by $x \cdot y = \sum_{i=1}^{n} x_i y_i$. We can now restate Theorem 7 as follows.

THEOREM 7 (Formal version). *The system*

$$\begin{aligned} \alpha_i \cdot x &> 0, \qquad i = 1, \ldots, m', \\ \beta_i \cdot x &= 0, \qquad i = 1, \ldots, m'' \end{aligned} \tag{1''}$$

has a solution iff the convex hull of $\alpha_1, \ldots, \alpha_{m'}$ *does not intersect the subspace generated by* $\mathbf{0}, \beta_1, \ldots, \beta_{m''}$. *That is, System* (1) *has a solution iff there is no solution* $\lambda_1, \ldots, \lambda_{m'}, \mu_1, \ldots, \mu_{m''}$ *to the system*

$$\begin{aligned} &\sum_{i=1}^{m'} \lambda_i \alpha_{ij} = \sum_{i=1}^{m''} \mu_i \beta_{ij}, \qquad j = 1, \ldots, n, \\ &\lambda_i \geqslant 0, \qquad i = 1, \ldots, m', \\ &\sum_{i=1}^{m'} \lambda_i = 1. \end{aligned} \tag{2}$$

The importance of this result depends partly on its constructive nature. One can start out to construct a solution to System (1″), but the procedure will eventually terminate in failure if and only if there is a solution to System (2).

An additional point is that the proof holds whether the numbers involved are real numbers or rationals. That is, if the α_{ij}, β_{ij} in System (1) are rational, then there is a rational solution to System (1) if and only if there is no rational solution to System (2). In System (2), the constraint $\sum \lambda_i = 1$ can just as well be replaced by $\sum \lambda_i > 0$, for then we could divide each value of λ_i and of μ_i by $\sum \lambda_i$ to obtain $\sum \lambda_i = 1$. Let the revised system, with $\sum \lambda_i > 0$, be numbered (2′). Note that a rational solution to System (2) yields, by multiplying through by a common denominator, an integer solution to System (2′). The actual criterion thus amounts to the nonexistence of integer solutions to System (2′). This is the basis for measurement axiomatizations (see Chapter 9).

The proof is based solely on the following well-known result of linear algebra, which we formulate as Lemma 7, but do not prove (see Gale, 1960).

LEMMA 7. *Let* $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{in})$, *and suppose that* $\alpha_1, \ldots, \alpha_m$ *are linearly independent (no one of them lies in the subspace generated by the other* $m - 1$). *Then for any numbers* $t_1, \ldots, t_m$ *the equations*

$$\alpha_i \cdot x = t_i, \qquad i = 1, \ldots, m$$

have a solution $x = (x_1, \ldots, x_n)$.

That is, it is always possible to solve m linear equations $\alpha_i \cdot x = t_i$ in n unknowns $x_1, \ldots, x_n$, provided that the m linear expressions $\sum \alpha_{ij} x_j$ are independent of one another and so, in particular, $m \leqslant n$. Of course, the claim that the rest of the proof of Theorem 7 is constructive is based on the existence of a constructive proof to Lemma 7. The construction of a solution $(x_1, \ldots, x_n)$ is well known however. One merely uses the first equation to express one of the x_i in terms of the other $n - 1$, substituting the expression for x_i in the other $m - 1$ equations, reducing the problem to $m - 1$ equations in $n - 1$ unknowns. This process continues until either there are no equations left—in which case one can assign arbitrary values to the remaining unknowns—or until one of the substitutions results in a contradiction, in which case the process stops and it can be shown that the original α_i were not independent.

The next theorem belongs to the theory of linear inequalities, and since it serves as a basis for measurement axiomatizations we present its proof in detail.

LEMMA 8. *Let* $\{\alpha_{ij} \mid i = 1, \ldots, m, j = 1, \ldots, n\}$ *be an* $m \times n$ *matrix,* $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{in})$, $\alpha^{(j)} = (\alpha_{1j}, \ldots, \alpha_{mj})$, *and* $z = (z_1, \ldots, z_n)$. *The system of inequalities*

$$\begin{aligned} \alpha_i \cdot x &\geqslant 0, \qquad i = 1, \ldots, m, \\ z \cdot x &< 0 \end{aligned} \tag{3}$$

has a solution $x = (x_1, \ldots, x_n)$ *iff the system*

$$y \cdot \alpha^{(j)} = z_j, \qquad j = 1, \ldots, n, \qquad y_i \geqslant 0, \qquad i = 1, \ldots, m \tag{4}$$

has no solution $y = (y_1, \ldots, y_m)$.

PROOF. Suppose that x is a solution to System (3) and y is a solution to System (4). Then

$$0 > z \cdot x = \sum_{j=1}^{n} z_j x_j = \sum_{j=1}^{n} [y \cdot \alpha^{(j)}]\, x_j = \sum_{i=1}^{m} y_i[\alpha_i \cdot x] \geqslant 0,$$

since y_i, $\alpha_i \cdot x \geqslant 0$, $i = 1, \ldots, m$. This is impossible.

Suppose that System (4) has no solution. We show that System (3) has a solution. First, suppose that the equations $y \cdot \alpha^{(j)} = z_j$ have no solution at all, i.e., z is not in the subspace generated by $\alpha_1, \ldots, \alpha_m$. Reindex the α_i so that $\alpha_1, \ldots, \alpha_r$ are an independent set and $\alpha_{r+1}, \ldots, \alpha_m$ are in the subspace generated by them. Then $\alpha_1, \ldots, \alpha_r, z$ are independent. By Lemma 7, we can find x such that $\alpha_i \cdot x = 0$, $i = 1, \ldots, r$ and $z \cdot x < 0$. Since this implies $\alpha_i \cdot x = 0$, $i = r + 1, \ldots, m$, x is a solution to (3).

Second, suppose that $y \cdot \alpha^{(j)} = z_j$, $j = 1, \ldots, n$, but that $y_i < 0$ for at least one i. We proceed by induction on m. If $m = 1$, then $y_1\alpha_{1j} = z_j$, $j = 1, \ldots, n$, and $y_1 < 0$. Let $x = \alpha_1$. Now $\alpha_1 \neq \mathbf{0}$, otherwise $z = \mathbf{0}$, and $y_1 = 0$ would solve (4). Thus, $\alpha_1 \cdot x > 0$, $z \cdot x = y_1(\alpha_1 \cdot x) < 0$, and x is a solution to System (3). Suppose now that Lemma 8 holds for $m \leqslant k$, where $k \geqslant 1$, and consider the case $m = k + 1$. Let $\beta^{(j)} = (\alpha_{1j}, \ldots, \alpha_{kj})$. There can be no solution to System (4) with $\beta^{(j)}$ substituted for $\alpha^{(j)}$, for if there were, we would let $y_{k+1} = 0$, obtaining a solution to the original system. By the inductive hypothesis, there is a vector $x' = (x_1', \ldots, x_n')$ with $\alpha_i \cdot x' \geqslant 0$, $i = 1, \ldots, k$ and $z \cdot x' < 0$. If $\alpha_{k+1} \cdot x' \geqslant 0$, then let $x = x'$ and we are done. So suppose that $\alpha_{k+1} \cdot x' < 0$. Define

$$\alpha_i' = (\alpha_i \cdot x')\,\alpha_{k+1} - (\alpha_{k+1} \cdot x')\,\alpha_i, \qquad i = 1, \ldots, k,$$
$$z' = (z \cdot x')\,\alpha_{k+1} - (\alpha_{k+1} \cdot x')z.$$

Consider the system

$$y' \cdot \alpha'^{(j)} = z_j', \qquad j = 1, \ldots, n, \qquad y_i' \geqslant 0, \qquad i = 1, \ldots, k \tag{4'}$$

where $\alpha'^{(j)} = (\alpha'_{1j}, \ldots, \alpha'_{kj})$, $\alpha_i' = (\alpha'_{i1}, \ldots, \alpha'_{in})$. If y' is a solution to System (4′), define y by

$$y_i = y_i', \qquad i = 1, \ldots, k$$

$$y_{k+1} = -(\alpha_{k+1} \cdot x')^{-1} \Big[\sum_{i=1}^{k} y_i'(\alpha_i \cdot x') - (z \cdot x')\Big].$$

We show that y is a solution to System (4). In fact, $y_i = y_i' \geqslant 0$, $i = 1, \ldots, k$, and $y_{k+1} \geqslant 0$ since $-(\alpha_{k+1} \cdot x')^{-1} > 0$, $y_i' \geqslant 0$, $\alpha_i \cdot x' \geqslant 0$, and $-(z \cdot x') > 0$. Moreover, for $j = 1, \ldots, n$,

$$\begin{aligned}
z_j &= -(\alpha_{k+1} \cdot x')^{-1}\,[z_j' - (z \cdot x')\,\alpha_{k+1,j}] \\
&= -(\alpha_{k+1} \cdot x')^{-1}\,[y' \cdot \alpha'^{(j)} - (z \cdot x')\,\alpha_{k+1,j}] \\
&= -(\alpha_{k+1} \cdot x')^{-1}\Big[\sum_{i=1}^{k} y_i'\,\alpha'_{ij} - (z \cdot x')\,\alpha_{k+1,j}\Big] \\
&= -(\alpha_{k+1} \cdot x')^{-1}\Big[\sum_{i=1}^{k} y_i'(\alpha_i \cdot x')\,\alpha_{k+1,j} \\
&\qquad - \sum_{i=1}^{k} y_i'(\alpha_{k+1} \cdot x')\,\alpha_{ij} - (z \cdot x')\,\alpha_{k+1,j}\Big] \\
&= \sum_{i=1}^{k} y_i'\alpha_{ij} + y_{k+1}\alpha_{k+1,j} \\
&= y \cdot \alpha^{(j)}.
\end{aligned}$$

Thus, y is a solution to System (4), contrary to hypothesis. We conclude that there exists no solution y' to System (4′). By the inductive hypothesis, there exists a solution $x'' = (x_1'', \ldots, x_n'')$ to the system

$$\alpha_i' \cdot x'' \geqslant 0, \qquad i = 1, \ldots, k, \qquad z' \cdot x'' < 0. \tag{3'}$$

Define

$$x = (\alpha_{k+1} \cdot x'')\,x' - (\alpha_{k+1} \cdot x')\,x''.$$

We show that x is a solution to System (3). In fact, it is easy to verify, from the definition of α_i', z', and x, that

$$\begin{aligned} \alpha_i \cdot x &= \alpha_i' \cdot x'' \geqslant 0, \qquad i = 1,..., k, \\ \alpha_{k+1} \cdot x &= 0, \\ z \cdot x &= z' \cdot x'' < 0. \end{aligned}$$

By induction, the theorem follows for any m. ◇

The preceding proof allows us to construct the solution to a system of inequalities (3) whenever the solution exists. First, obtain a solution $x^{(1)}$ to $\alpha_1 \cdot x \geqslant 0, z \cdot x < 0$. If the equations $\alpha_1 \cdot x = 0$, $z \cdot x = -1$ have a simultaneous solution, then it will serve as $x^{(1)}$. If they do not, then $z = \lambda\alpha_1$ for some λ. If $\lambda \geqslant 0$, then no solution is possible, since $\alpha_1 \cdot x \geqslant 0$ implies $z \cdot x \geqslant 0$. If $\lambda < 0$, then choose $x^{(1)} = \alpha_1$. Now check the solution $x^{(1)}$ in successive inequalities $\alpha_2 \cdot x \geqslant 0$,... until, for some $k \geqslant 1$, $\alpha_{k+1} \cdot x^{(1)} < 0$. At this point, $x^{(1)}$ solves the first k inequalities $\alpha_i \cdot x \geqslant 0$, as well as $z \cdot x < 0$, so relabel $x^{(1)}$ as $x^{(k)}$. To construct $x^{(k+1)}$, proceed as in the inductive step to the above proof, defining α_i', $i = 1,..., k$ and z', and solving the system $\alpha_i' \cdot x'' \geqslant 0$, $z' \cdot x'' < 0$. Since this system has fewer inequalities than System (3), a genuine reduction is achieved, although the method here described may have to be applied in full panoply to this lesser system. If the reduced system has no solution, then, as the lemma shows, the full system also has no solution. If x'' solves the reduced system, define $x^{(k+1)}$ from $x' = x^{(k)}$ and x'' as in the lemma. Now check $x^{(k+1)}$ in successive inequalities $\alpha_i \cdot x \geqslant 0$, $i \geqslant k + 2$, until it fails; then apply the method of the inductive step over again, etc. (see Exercise 15).

2.3.3 Proof of Theorem 7

First we show that Systems (1″) and (2) cannot both have solutions. For if they did, we would have

$$0 < \sum_{i=1}^{m'} \lambda_i(\alpha_i \cdot x) = \sum_{i=1}^{m''} \mu_i(\beta_i \cdot x) = 0.$$

Next, suppose that System (1″) has no solution. Define vectors γ_i in Re^{n+1} by

$$\begin{aligned} \gamma_i &= (\alpha_{i1} ,..., \alpha_{in} , 1), & i &= 1,..., m', \\ \gamma_{m'+i} &= (\beta_{i1} ,..., \beta_{in} , 0), & i &= 1,..., m'', \\ \gamma_{m'+m''+i} &= (-\beta_{i1} ,..., -\beta_{in} , 0), & i &= 1,..., m''. \end{aligned}$$

Let $z_j = 0, j = 1,..., n, z_{n+1} = 1$. If System (1″) has no solution, then neither does the system

$$\begin{aligned} \gamma_i \cdot \bar{x} &\geqslant 0, \qquad i = 1,..., m' + 2m'', \\ z \cdot \bar{x} &< 0. \end{aligned} \tag{5}$$

For if System (5) had a solution $\bar{x} = (x_1 ,..., x_{n+1})$, $z \cdot \bar{x} < 0$ would imply $x_{n+1} < 0$. Thus, setting $x = (x_1 ,..., x_n)$, we have from $\gamma_i \cdot \bar{x} \geqslant 0$ the results

$$\begin{aligned} \alpha_i \cdot x + x_{n+1} &\geqslant 0, & i &= 1,..., m', \\ \beta_i \cdot x &\geqslant 0, & i &= 1,..., m'', \\ -\beta_i \cdot x &\geqslant 0, & i &= 1,..., m''. \end{aligned}$$

Thus, $\alpha_i \cdot x > 0, \beta_i \cdot x = 0$ as required in System (1″). Let $\gamma^{(j)} = (\gamma_{1j} ,..., \gamma_{mj})$, where $m = m' + 2m''$. By Lemma 8, the system

$$y \cdot \gamma^{(j)} = z_j , \qquad y_i \geqslant 0$$

has a solution. Let $\lambda_i = y_i$, $i = 1,..., m'$ and let $\mu_i = -y_{m'+i} + y_{m'+m''+i}$, $i = 1,..., m''$. Then

$$\begin{aligned} \sum_{i=1}^{m'} \lambda_i\alpha_{ij} - \sum_{i=1}^{m''} \mu_i\beta_{ij} &= y \cdot \gamma^{(j)} \\ &= z_j \\ &= 0, \qquad j = 1,..., n, \end{aligned}$$

and

$$\sum_{i=1}^{m'} \lambda_i = y \cdot \gamma^{(n+1)} = z_{n+1} = 1.$$

Thus, there is a solution to System (2). ◇

Note that this proof leads to a construction of solutions to (1), by solving the associated System (5) using the method of Lemma 8. More efficient computational methods for solving System (5) are available, however, via linear programming (see Gale, 1960).

2.3.4 Topological Proof of Theorem 7

In this section, we give a shorter, more intuitive, but nonconstructive proof of Theorem 7, which fails if the base field is Ra rather than Re.

First, consider the case where there are no equalities in System (1), i.e., $m'' = 0$. In this case, the theorem states that a solution to System (1) exists if and only if the convex hull K of $\{\alpha_1, ..., \alpha_{m'}\}$ does not intersect $\mathbf{0}$. The set K is closed and bounded in Re^n, hence, it is compact. For $y \in K$, $y \cdot y$ (the squared distance of y from $\mathbf{0}$) is a continuous function of y, and thus is minimal for some $x \in K$. By convexity, x is unique, but this result is not needed. (See Rudin, 1964, Sections 2.41, 4.11, 4.16 for the details of the argument just given.) We claim that x is a solution to System (1). If not, then for some i, $\alpha_i \cdot x \leqslant 0$. For $\lambda \in \mathrm{Re}$, define

$$f(\lambda) = [\lambda\alpha_i + (1 - \lambda) x] \cdot [\lambda\alpha_i + (1 - \lambda) x].$$

Differentiating f with respect to λ and setting the derivative equal to zero yields that f is minimum at

$$\lambda_0 = (x \cdot x - \alpha_i \cdot x)/(\alpha_i - x) \cdot (\alpha_i - x).$$

Since $x \cdot x$ and $(\alpha_i - x) \cdot (\alpha_i - x)$ are greater than 0 and $-\alpha_i \cdot x \geqslant 0$, $\lambda_0 > 0$. It is easily checked that $1 > \lambda_0$. Let $y = \lambda_0\alpha_i + (1 - \lambda_0)x$. Then $y \in K$, $y \neq x$, and

$$y \cdot y = f(\lambda_0) < f(0) = x \cdot x,$$

contradicting minimality of $x \cdot x$. Therefore, $\alpha_i \cdot x > 0$ for $i = 1, ..., m'$.

Next, suppose that $m'' > 0$. Let B be the subspace generated by $\beta_1, ..., \beta_{m''}$ and let C be its orthogonal complement (see MacLane & Birkhoff, 1967, p. 240). Let C be generated by vectors $\gamma_1, ..., \gamma_k$. Define new vectors α_i' by

$$\alpha_{ij}' = \alpha_i \cdot \gamma_j, \qquad i = 1, ..., m', \quad j = 1, ..., k.$$

If x is any solution to System (1), then $x \in C$, so $x = \sum_{j=1}^{k} x_j'\gamma_j$, for some $x' = (x_1', ..., x_k')$. Thus,

$$\begin{aligned} \alpha_i' \cdot x' &= \sum_{j=1}^{k} (\alpha_i \cdot \gamma_j) x_j' \\ &= \alpha_i \cdot x > 0, \qquad i = 1, ..., m'. \end{aligned}$$

Conversely, if x' is a solution of the system

$$\alpha_i' \cdot x' > 0, \qquad i = 1, ..., m', \tag{6}$$

then $x = \sum_{j=1}^{k} x_j'\gamma_j$ is a solution to System (1). Thus we can apply the criterion derived above for $m'' = 0$ to the reduced System (6). In fact, the convex hull of the α_i' intersects the origin (of Re^k) if and only if there exist $\lambda_1, ..., \lambda_{m'} \geqslant 0$, with $\sum_{i=1}^{m'} \lambda_i = 1$, such that for $j = 1, ..., k$,

$$\begin{aligned} 0 &= \sum_{i=1}^{m'} \lambda_i \alpha_{ij}' \\ &= \sum_{i=1}^{m'} \lambda_i (\alpha_i \cdot \gamma_j) \\ &= \left[\sum_{i=1}^{m'} \lambda_i \alpha_i\right] \cdot \gamma_j . \end{aligned}$$

But this last equation means that $[\sum_{i=1}^{m'} \lambda_i\alpha_i] \cdot x = 0$ for every $x \in C$. A well-known result on orthogonal complements implies that $\sum_{i=1}^{m'} \lambda_i\alpha_i \in B$, and this is precisely the criterion for nonsolution of System (1). ◇

EXERCISES

1. Carry out the inductive step in the proof of Theorem 1, showing that if ϕ is order preserving on $\{a_1, ..., a_n\}$, then it is also order preserving on $\{a_1, ..., a_{n+1}\}$. (2.1)

2. If you did not do Exercise 1.5, this would be a good time to do it.

3. Prove Lemma 2. (2.2.3)

In the following nine exercises, $\langle A, \succsim, \oplus, * \rangle$ is an Archimedean ordered ring (Definition 5) with additive identity (zero) θ. Let $(n - 1)\, a \oplus a = na$, $a^{n-1} * a = a^n$, and let $-a$ be the additive inverse of a. (2.2.7)

4. Prove the following. If ϕ is order preserving and multiplicative, then for any $a \succ \theta$, the following are equivalent:

(i) $\phi(a) < 1$,
(ii) $a \succ a^2$.

5. Prove that if ϕ is order preserving, additive, and multiplicative, then for any $a \succ \theta$ and any $m, n > 0$, the following are equivalent:

(i) $\phi(a) < m/n$,
(ii) $ma \succ na^2$.

6. Prove that for any $a \succ \theta$, the set of $m, n > 0$ such that $ma \succ na^2$ is nonempty.

Now *define* ϕ for $a \succ \theta$ by using Exercises 5 and 6:

$$\phi(a) = \inf\{m/n \mid m, n > 0 \quad \text{and} \quad ma \succ na^2\}.$$

The next six exercises show that ϕ is order preserving, additive, and multiplicative on $A^+ = \{a \mid a \succ \theta\}$, and hence, that ϕ yields an isomorphism into the ordered ring of real numbers. All a, b, c below are understood to be $\succ \theta$.

7. $m(a * b) = (ma) * b = a * (mb)$.

8. If $ma \succ na^2$, then for all b,

(i) $mb \succ na * b$,
(ii) $mb \succ nb * a$.

9. Suppose $a \succsim b$. If $ma \succ na^2$, then $mb \succ nb^2$; hence, $\phi(a) \geqslant \phi(b)$.

10. If $ma \succ na^2, m'b \succ n'b^2$, then $(mn' + m'n)(a \oplus b) \succ nn'(a \oplus b)^2$. Hence, $\phi(a \oplus b) \leqslant \phi(a) + \phi(b)$.

11. If $m/n < \phi(a)$, then $ma \prec na^2$. Hence, $0 < \phi(a) = \sup\{m/n \mid ma \prec na^2\}$, $\phi(a \oplus b) = \phi(a) + \phi(b)$, and $a \succ b$ implies $\phi(a) > \phi(b)$.

12. Prove that ϕ is multiplicative.

13. Modify Exercises 4–12 so as to generate a proof of Theorem 6 for ordered semirings. (2.2.7)

14. In the proof of the corollary to Theorem 6 prove that the restriction of $\succsim, \oplus, *$ to A^+ forms an Archimedean, positive, regular, ordered, local semiring. (2.2.7)

15. Use the method of Section 2.3.2 to solve the system

$$x_1 + x_2 \geqslant 0,$$
$$2x_1 + 3x_2 \geqslant 0,$$
$$2x_1 + 5x_2 < 0.$$

Chapter 3 Extensive Measurement

□ 3.1 INTRODUCTION

Extensive attributes such as length and mass have been measured successfully since antiquity. The modern theory of extensive measurement, however, originated less than a century ago when Helmholtz (1887) and Hölder (1901) developed the first axiomatic analysis of extensive attributes. Generally speaking, a theory of extensive measurement is a set of assumptions, or axioms, formulated in terms of an ordering $\succsim$ (of objects with respect to some property) and a concatenation operation $\circ$ (between objects) that permit the construction of a scale ϕ satisfying

(i) $a \succsim b$ iff $\phi(a) \geqslant \phi(b)$,
(ii) $\phi(a \circ b) = \phi(a) + \phi(b)$.

Since the additive representation of mass, length, and time duration have become a part of our daily experience we tend to take for granted the qualitative laws (e.g., $a \succsim b$ whenever $a \circ c \succsim b \circ c$) that underlie the numerical representation. Indeed, under the natural interpretations of $\succsim$ and $\circ$, these laws typically reduce to common physical truths. Nevertheless, the formulation of a set of axioms that are sufficient for the representation and acceptable from an empirical standpoint poses several problems to which the present chapter is devoted.

Several attempts to improve Hölder's (1901) theory have been made.