Measurement Inequalities and Dimensional Analysi

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Measurement Inequalities and Dimensional Analysis



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Foundations of Measurement Spring 2011

IF YOU KEEP SAYING "BEAR WITH ME FOR A MOMENT", PEOPLE TAKE A WHILE TO FIGURE OUT THAT YOU'RE JUST SHOWING THEM RANDOM SLIDES.

2011-04-25

-Outline

Outline

oasurement Inequalities Solvability Finite Linear Structures Polynomial Structures

Dimensional Analogis Phopical Laws The Algebra of Phopical Quantities The Pi Theorem and Dimensional Analogis Examples of Dimensional Analogis Consistency of Dorived Measures Embedding into a Structure of Phopical Quantities Why are Numerical Laws Dimensionality Invariant?

1. About 1 hour on Measurement Inequalities, then a break

2. Remainder of the time on Dimensional Analysis

Outline

Measurement Inequalities

Solvability Finite Linear Structures Polynomial Structures

Dimensional Analysis

Physical Laws The Algebra of Physical Quantities The Pi Theorem and Dimensional Analysis Examples of Dimensional Analysis Consistency of Derived Measures Embedding into a Structure of Physical Quantities Why are Numerical Laws Dimensionally Invariant?



(transition)

Outline

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Solvabilitu

Measurement Inequalities Solvability Finite Linear Structures Polynomial Structures

Dimensional Analysis

Physical Laws The Algebra of Physical Quantities The Pi Theorem and Dimensional Analysis Examples of Dimensional Analysis Consistency of Derived Measures Embedding into a Structure of Physical Quantities Why are Numerical Laws Dimensionally Invariant?

Solvability Conditions

Examples

If a > b, then there exists d ∈ A such that (b, d) ∈ B and a ²/_L b < d. (Bi)
 If ab, cd ∈ A^{*} and ab > cd, then there exist d^{*}, d^{*} ∈ A such that ad^{*}, d^{*}b, ad^{*}, d^{*}b ∈ A^{*} and ad^{*} ~ cd ~ d^{*}b. (147)
 Definition 6.5 (266)
 Most Closure axioms.

1. So far, we have made heavy use of various solvability axioms.

Solvability Conditions

Examples

- If $a \succ b$, then there exists $d \in A$ such that $(b, d) \in B$ and $a \succeq b \circ d$. (84)
- If ab, $cd \in A^*$ and $ab \succ cd$, then there exist d', $d'' \in A$ such that ad', d'b, ad'', $d''b \in A^*$ and $ad' \sim cd \sim d''b$. (147)
- Definition 6.5 (256)
- Most Closure axioms.



Failures of Solvability

 Solubility axioms are existence claims, so they are usually non-necessary.
 There are models that almost saridy the conjoint measurement furcture, for instance, but nor variable is discorte and the other is not equally spaced.
 Useruit solubility is a sale assumption, the shape of the data can make solving the requisite equations practically impossible.
 Useruit a solvability doublitity condition is true in the

Even it a nontested solvability condition is true in the underliging data-generating process and if the tosted necessary conditions are true in the obtained factorial data, it does not follow that the obtained data possess a representation of the kind in question. (425)

- 1. Solvability claims are usually non-necessary
- 2. Some models can get close to being, for example, a conjoint measurement structure, but they slightly miss.
- 3. Even if data generating processes satisfy solvability, that does not mean that the data collected also satisfy it, nor does it mean that the equations implied by the data are practically solvable.

Failures of Solvability

- Solvability axioms are existence claims, so they are usually non-necessary.
- There are models that almost satisfy the conjoint measurement structure, for instance, but one variable is discrete and the other is not equally spaced.
- Even if solvability is a safe assumption, the shape of the data can make solving the requisite equations practically impossible.
- Even if a nontested solvability condition is true in the underlying data-generating process and if the tested necessary conditions are true in the obtained factorial data, it does not follow that the obtained data possess a representation of the kind in question. (425)



- 1. Suppose we have a $3\times3\times2$ factorial design and we get the ordinal data shown in the top two tables
- 2. These data do not violate the independence axioms necessary for an additive decomposition. An example of one of the checks is in the bottom left table.
- 3. However, these data do not satisfy double cancellation (inequalities are schematic).
- 4. Since solvability and independence imply double cancellation, the data generated cannot satisfy solvability.
- 5. So what can we do if we can't assume solvability?

Failures of Solvability

Example

Ex

b1 a1 b1 c1

 $a_1 + b_2 \le b_1 + c_2$

am	ple	from	425							
a ₃					b_3					
		a_1	b_1	c_1			a_1	b_1	<i>c</i> ₁	
	<i>a</i> 2	12	14	18		a ₂	11	13	17	
	b_2	4	10	16		b_2	3	9	15	
	<i>c</i> ₂	2	6	8		<i>c</i> ₂	1	5	7	
	a	b ₃								
a ₂	6	5			$a_1b_2\precsim b_1c_2$		ā	$b_1 + k_1$	$b_2 \leq l$	$b_1 + c_2$
b_2	4	3			$b_1a_2\precsim c_1b_2$		Ŀ	$p_1 + a_2$	$a_2 \leq a_2$	$c_1 + b_2$
с2	2	1			$a_1a_2 \succ c_1c_2$		ê	$a_1 + a_2$	$a_2 > a_2$	$c_1 + c_2$



(transition)

Outline

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Finite Linear Structures

Measurement Inequalities Solvability Finite Linear Structures Polynomial Structures

Dimensional Analysis

Physical Laws The Algebra of Physical Quantities The Pi Theorem and Dimensional Analysis Examples of Dimensional Analysis Consistency of Derived Measures Embedding into a Structure of Physical Quantities Why are Numerical Laws Dimensionally Invariant?

Measurement Inequalities
Finite Linear Structures
Finite Linear Structures

Finite Linear Structures

Additive Conjoint Mode

$$\begin{split} & \text{Suppose } A_{i} \text{ and } A_{i} \text{ are limits sets.} \\ & \text{Is } I_{i} \gtrsim be a weak action of A - A_{i} \times A_{i}. \\ & \text{We want Is fill encoreasing and sufficient conditions such that A^{i} \gtrsim [a^{i} H^{i}_{i} \cap (j) + c_{i}(j)) > c_{i}(j) + c_{i}(j). \\ & \text{This is possible are further number of A_{i} (are that all e-th-order concellation axioms hold. \\ & \text{Furthermore, all in the order concellation axioms were implied by independence, double-concellation, Archimedean-mess, and restricted subshifts. \end{split}$$

Finite Linear Structures

Additive Conjoint Models

- Suppose *A*₁ and *A*₂ are finite sets.
- Let \succeq be a weak order of $A = A_1 \times A_2$.
- We want to find necessary and sufficient conditions such that $ap \succeq bq$ iff $\phi_1(a) + \phi_2(p) \ge \phi_1(b) + \phi_2(q)$.
- This is possible for any finite number of A_i given that all n-th order cancellation axioms hold.
- Furthermore, all *n*-th order cancellation axioms were implied by independence, double-cancellation, Archimedean-ness, and restricted solvability.

1. (read slide)

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2. Since we want additive representations without solvability, we need something else to get us all of the cancellation axioms.

-Finite Linear Structures

2011-04-25 -Measurement Inequalities Finite Linear Structures

Finite Linear Structures Andliary Space Construction

• Let $A = A_1 \times \ldots \times A_n$. Suppose |A_i| = k_i < ∞ for all i and A_i ∩ A_i = ∅ for all i ≠ i. ■ Let > be a reflexive binary relation on A (think a weak) ordering, but it doesn't need to be transitive or connected). • Let $k = \sum_{i=1}^{n} k_i$ be the size of $Y = \bigcup_{i=1}^{n} A_i$. Enumerate the elements of Y as y₁,..., y_k. Define an injective mapping v: A → ℝ^k by a → 𝔅 = (𝔅₂,...,𝔅_k), $\overline{a}_{i} = \begin{cases} 1 & \text{if } y_{i} \text{ is a component of } a \\ 0 & \text{otherwise} \end{cases}$

Finite Linear Structures

Auxiliary Space Construction

- Let $A = A_1 \times \ldots \times A_n$.
- Suppose $|A_i| = k_i < \infty$ for all *i* and $A_i \cap A_i = \emptyset$ for all $i \neq j$.
- Let \succeq be a reflexive binary relation on *A* (think a weak ordering, but it doesn't need to be transitive or connected).

• Let
$$k = \sum_{i=1}^{n} k_i$$
 be the size of $Y = \bigcup_{i=1}^{n} A_i$.

- Enumerate the elements of *Y* as y_1, \ldots, y_k .
- Define an injective mapping $v: A \to \mathbb{R}^k$ by $a \mapsto \overline{a} = (\overline{a}_1, \dots, \overline{a}_k)$, where

$$\overline{a}_{i} = \begin{cases} 1 & \text{if } y_{i} \text{ is a component of } a \\ 0 & \text{otherwise} \end{cases}$$



Finite Linear Structures Auditary Space Construction Continued

 $\begin{array}{l} \mathbf{t} \in \overline{A} = \{\overline{A} \in A\} \text{ and } A^+ \text{ be the addition chosen on } \overline{A} \\ = \mathbf{b}(\mathbf{t} \mathbf{t} \sim (\mathbf{a} \wedge \mathbf{t}) \mathbf{s} \sim (\mathbf{y} + \mathbf{t}) \mathbf{s} + \mathbf{s} - \mathbf{s} \\ \mathbf{y}^{(0)}, \dots, \mathbf{y}^{(0)} \in \overline{B}^{(0)}, \dots, \overline{B}^{(0)} \in \overline{A} \text{ and } \\ \mathbf{y} = \sum_{i=1}^{N} \overline{B}^{(0)} \text{ and } \mathbf{y}^{(i)} \sim \overline{B}^{(0)} \text{ cond } \mathbf{s}^{(i)} \\ = \mathbf{b}(\mathbf{t} \mathbf{s} \sim (\mathbf{s} \wedge \mathbf{t}) \mathbf{s} - \mathbf{s} - \mathbf{s} - \mathbf{s} - \mathbf{s} \\ \mathbf{y}^{(i)}, \dots, \mathbf{s}^{(i)} \in \overline{B}^{(i)}, \dots, \overline{B}^{(i)} \in \overline{A} \text{ and } \mathbf{b} \mathbf{t} \mathbf{s} - \mathbf{s} \\ = \overline{B}^{(i)}, \dots, \overline{B}^{(i)}, \dots, \overline{B}^{(i)} \in \overline{B} \text{ both } \mathbf{b} \mathbf{t} \mathbf{s} - \mathbf{s} \\ = \overline{B}^{(i)}, \dots, \mathbf{s}^{(i)} \in \overline{B} \text{ both } \mathbf{s} + \mathbf{s} \text{ and } \mathbf{b} \mathbf{t} \mathbf{s} - \mathbf{s} \\ = \overline{B}^{(i)}, \dots, \overline{B}^{(i)} \in \overline{B} \text{ both } \mathbf{s} + \mathbf{s} \text{ and } \mathbf{b} \mathbf{t} \mathbf{s} - \mathbf{s} \\ = \overline{B}^{(i)}, \mathbf{s} = \overline{B}^{(i)}, \mathbf{s} = \overline{B}^{(i)}, \mathbf{s} \in \overline{B} \text{ both } \mathbf{s} + \mathbf{s} \text{ and } \mathbf{s} \text{ cons } \mu, \overline{B}, \overline{B} \geq \mathbf{y} 0. \\ = \mathbf{D} \text{ and } \mathbf{s}_{i}^{(i)} \text{ sup } \mathbf{s}^{(i)} = \mathbf{s}_{i}^{(i)} + \mathbf{s} \end{array}$

Finite Linear Structures

Auxiliary Space Construction Continued

- Let $\overline{A} = {\overline{a} | a \in A}$ and A^+ be the additive closure over \overline{A} .
- Define \sim_{I} on A^{+} by $x \sim_{I} y$ iff there are $\overline{a}^{(1)}, \dots, \overline{a}^{(m)}, \overline{b}^{(1)}, \dots, \overline{b}^{(m)} \in \overline{A}$ such that $x = \sum_{i=1}^{m} \overline{a}^{(i)}$ and $y = \sum_{i=1}^{m} \overline{b}^{(i)}$ and $\overline{a}^{(i)} \sim \overline{b}^{(i)}$ for all *i*.
- Define \succ_{I} on A^{+} by $x \succ_{I} y$ iff there are $\overline{a}^{(1)}, \dots, \overline{a}^{(m)}, \overline{b}^{(1)}, \dots, \overline{b}^{(m)} \in \overline{A}$ such that $x = \sum_{i=1}^{m} \overline{a}^{(i)}$ and $y = \sum_{i=1}^{m} \overline{b}^{(i)}$ and $\overline{a}^{(i)} \succeq \overline{b}^{(i)}$ for all *i* and for some *j*, $\overline{b}^{(j)} \succeq \overline{a}^{(j)}$.
- Define \succeq_I on A^+ as $\succeq_I = \sim_I \cup \succ_I$.



Finite Linear Structures Properties of ~₂, >₂, and ≥₁

 The relation ~_I is reflexive and symmetric.
 The relation >_I is not necessarily irreflexive nor asymmetric (contrary to what the usual parallel with > might suggest).

Finite Linear Structures

Properties of \sim_I , \succ_I , and \succeq_I

- The relation \sim_I is reflexive and symmetric.
- The relation ≻_I is not necessarily irreflexive nor asymmetric (contrary to what the usual parallel with > might suggest).

Finite Linear Structures

-Finite Linear Structures

2011-04-25 -Measurement Inequalities Finite Linear Structures

Example Suppose \succeq on $A_1 \times ... \times A_n$ violates independence. So we have the following for some a. b. a'. b':

> $a = a_1 \cdots a_i \cdots a_n \succ b_1 \cdots a_i \cdots b_n = b'$ $b = b_1 \cdots b_r \cdots b_n \succ a_1 \cdots b_r \cdots a_n = a'$

Finite Linear Structures

Counterexample to \succ_I being necessarily asymmetric.

Example

Suppose \succeq on $A_1 \times \ldots \times A_n$ violates independence. So we have the following for some a, b, a', b':

$$a = a_1 \cdots a_i \cdots a_n \succ b_1 \cdots a_i \cdots b_n = b'$$

$$b = b_1 \cdots b_i \cdots b_n \succeq a_1 \cdots b_i \cdots a_n = a'$$



Exa	nple					
Now	, we hav	e, WLOG:				
	7-1	1,0,,0	,1,	0,,0,.	, 1, 0, .	
		Ă		Ă	, A	
	$\overline{x} = 0$	(1,0,,0	,0,	1, 0, , (1, , 1, 0	0,,0)
				Ă	· ·	Ă
	$\overline{b} = 1$	0.1.0	0	0.1.0		0.1.00
			~ `		~ ``	
	$\overline{N} = 1$	0.1.0	0	10 0	1 01	
		<u> </u>	~ `	<u> </u>	· ~	~

Finite Linear Structures

Counterexample to \succ_I being necessarily asymmetric.

Example

Now, we have, WLOG:



¹⁰ Finite Linear Structures

Example Adding $\overline{a} + \overline{b}$ and $\overline{b'} + \overline{a'}$, we get:

 $\overline{\sigma} + \overline{b} = (\underbrace{1, 1, 0, \dots, 0}_{A_1}, \underbrace{1, 1, 0, \dots, 0}_{A_r}, \underbrace{1, 1, 0, \dots, 0}_{A_r}) = \overline{b}^r + \overline{a}^r$ We have $\overline{a} + \overline{b} >_I \overline{b}^r + \overline{a}^r$ since we have:

 $a > b' \implies a \gtrsim b'$ $b \gtrsim a'$ $a > b' \implies b' \gtrsim a$ But we also have $\overline{b'} + \overline{a'} > \overline{a} + \overline{b'}$ since we have constitut of sum.

Finite Linear Structures

Counterexample to \succ_I being necessarily asymmetric.

Example

Adding $\overline{a} + \overline{b}$ and $\overline{b'} + \overline{a'}$, we get:



We have $\overline{a} + \overline{b} \succ_I \overline{b'} + \overline{a'}$ since we have:

$$\begin{array}{c} \mathbf{a} \succ \mathbf{b}' \implies \mathbf{a} \succsim \mathbf{b}' \\ \mathbf{b} \succsim \mathbf{a}' \\ \mathbf{a} \succ \mathbf{b}' \implies \mathbf{b}' \nsucceq \mathbf{a} \end{array}$$

But we also have $\overline{b'} + \overline{a'} \succ_I \overline{a} + \overline{b}$ since we have equality of sum.

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Finite Linear Structures

The example before shows on that irreflexivity of >₁ implies independence of 2... Similarly it can be shown that irreflexivity of >₁ implies every a distribution axion. >> and >₂ are the asymmetric irreflexivity of >₁ is the distribution axion. >> and >₂ are the asymmetric. When distribution axion is a distribution axion is a distribution axion axion is a distribution axion axion

Finite Linear Structures Moral of the Story

- The example before shows us that irreflexivity of ≻₁ implies independence of ≿.
- Similarly, it can be shown that irreflexivity of ≻₁ implies every *n*-th order cancellation axiom.
- Furthermore, ≻_I is irreflexive iff ≻_I and ∼_I are the asymmetric and symmetric parts of ≿_I respectively.
- Irreflexivity of \succ_I also implies that \succeq_I has no intransitive cycles, but does not imply that \succeq_I is in fact transitive.

G ← Measurement Inequalities Finite Linear Structures Finite Linear Structures Finite Linear Structures Meral of the Story: Representation Theorem

$$\begin{split} & \text{Theorem 1} \\ & \text{The relation } r_{2} \text{ is irreflexive iff there exist } \phi \colon Y \to \mathbb{R} \text{ and } \psi \colon A \to \mathbb{R} \text{ such that for all } a, b \in A: \\ & (i) \quad \psi(a) = \psi(a_{1}, \ldots, a_{b}) = \frac{\beta_{a}}{\beta_{a}} \phi(a_{1}) \\ & (i) \quad a - b \text{ inplies } \psi(a) = \psi(b) \\ & (iii) \quad a > b \text{ inplies } \psi(a) > \psi(b) \\ & (iii) \quad a > b \text{ inplies } \psi(a) > \psi(b) \end{split}$$

Finite Linear Structures

Moral of the Story: Representation Theorem

Theorem 1

The relation \succ_I is irreflexive iff there exist $\phi: Y \to \mathbb{R}$ and $\psi: A \to \mathbb{R}$ such that for all $a, b \in A$:

(i)
$$\psi(a) = \psi(a_1, \dots, a_n) = \sum_{i=1}^n \phi(a_i)$$

(ii) $a \sim b$ implies $\psi(a) = \psi(b)$
(iii) $a \succ b$ implies $\psi(a) > \psi(b)$

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-Measurement Inequalities -Finite Linear Structures

—Finite Linear Structures

Finite Linear Structures Meral of the Story: Representation Theorem

> Theorem 1 Proof Technique Theorem 1 can be proved by demonstrating the existence of a vector $z \in \mathbb{R}^3$ such that: (i) a > b implies $z \cdot 3 = z \cdot 5$ (ii) a > b implies $z \cdot 3 = z \cdot 5$ Then define $\phi(y_i) = z_i$.

Finite Linear Structures

Moral of the Story: Representation Theorem

Theorem 1 Proof Technique

Theorem 1 can be proved by demonstrating the existence of a vector $z \in \mathbb{R}^k$ such that:

(i)
$$a \sim b$$
 implies $z \cdot \overline{a} = z \cdot \overline{b}$

(ii)
$$a \succ b$$
 implies $z \cdot \overline{a} > z \cdot \overline{b}$

Then define $\phi(y_i) = z_i$.

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Finite Linear Structures Meral of the Story: Scale Type

> **Theorem 2** Suppose A has an order-preserving additive representation. Then there are vectors $x^{(j)}, \dots, x^{(m)} \in \mathbb{R}^{k}$ and an integer j with $0 \le j \le m$ such that z is an additive representation of A iff $x = \sum_{j=1}^{m} \alpha_{j} x^{(j)} + c$

 $x = \sum_{i=1}^{r} \alpha_i x^{ri} + c$

where $c = \lambda \vec{1}, \alpha_i \ge 0$ for $i \le j$, and $\alpha_i > 0$ for i > j. The representation is an interval scale iff m = 1.

Finite Linear Structures

Moral of the Story: Scale Type

Theorem 2

Suppose *A* has an order-preserving additive representation. Then there are vectors $z^{(1)}, \ldots, z^{(m)} \in \mathbb{R}^k$ and an integer *j* with $0 \le j \le m$ such that *z* is an additive representation of *A* iff

$$z = \sum_{i=1}^{m} \alpha_i z^{(i)} + c$$

where $c = \lambda \vec{1}$, $\alpha_i \ge 0$ for $i \le j$, and $\alpha_i > 0$ for i > j.

The representation is an interval scale iff m = 1.

└──Measurement Inequalities └──Finite Linear Structures └──Finite Linear Structures

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Finite Linear Structures Before and After

Before

 Independence, double-cancellation, Archimedean-ness, and restricted solvability imply all n-th order cancellations.
 All n-th order cancellations imply additive representation.

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Irreflexivity of >₁ implies independence and all n-th order cancellations.
 All n-th order cancellations imply additive representation.

Finite Linear Structures

Before and After

Before

- Independence, double-cancellation, Archimedean-ness, and restricted solvability imply all *n*-th order cancellations.
- All *n*-th order cancellations imply additive representation.

After

- Irreflexivity of ≻₁ implies independence and all *n*-th order cancellations.
- All *n*-th order cancellations imply additive representation.



Probability Structures

 We can do much the same thing for finite probability structures as well.
 Let X be a finite non-empty set, and let d be an algebra of sets on X, interpreted as events.
■ As before, define \$\vec{\vec{\vec{\vec{\vec{\vec{\vec{
Let z be a representation given by Theorem 1.
= Define $P(A) = \frac{z \cdot \overline{A}}{z \cdot \overline{X}}$
, and note that this satisfies all the requirements of probabilities. (433)
= This representation is possible iff \succ_I is irreflexive (Theorem 3).

Probability Structures

- We can do much the same thing for finite probability structures as well.
- Let X be a finite non-empty set, and let & be an algebra of sets on X, interpreted as events.
- As before, define $\overline{\mathscr{E}}$ and \mathscr{E}^+ and \sim_I, \succ_I with $\overline{\mathscr{E}}$ and \mathscr{E}^+ taking the place of \overline{A} and A^+ respectively.
- Let *z* be a representation given by Theorem 1.
- Define

$$P(A) = \frac{z \cdot \overline{A}}{z \cdot \overline{X}}$$

, and note that this satisfies all the requirements of probabilities. (433)

• This representation is possible iff \succ_I is irreflexive (Theorem 3).



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Outline

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Polunomial Structures

Measurement Inequalities

Solvability Finite Linear Structures Polynomial Structures

Dimensional Analysis

Physical Laws The Algebra of Physical Quantities The Pi Theorem and Dimensional Analysis Examples of Dimensional Analysis Consistency of Derived Measures Embedding into a Structure of Physical Quantities Why are Numerical Laws Dimensionally Invariant?

Polynomial Structures

 $\label{eq:second} \begin{array}{l} \bullet \mbox{ operator large of a large parameter model} \\ \bullet \mbox{ operator large of a polynomial insequence large parameters}. \\ \bullet \mbox{ There is a map from } a_1 \dots a_n \in A_1 \times \dots \times A_n$ to a polynomial parameter large parameters in the unknown second responsible, there are not a second responsible. The second response has a second response to the second response has a second response between the second response has a second response ha

unknowns is still designated Y. = Dofine the relation Σ_j on the set of polynomials corresponding to some $a_1 \cdots a_n$ such that when p corresponds to $a_1 \cdots a_n$ and q to $b_1 \cdots b_m$, we have $p \ge q$ iff $a_1 \cdots a_n \ge b_1 \cdots b_n$.

Polynomial Structures

- In general, factorial data and a proposed measurement model give rise to a set of polynomial inequalities.
- There is a map from $a_1 \cdots a_n \in A_1 \times \cdots \times A_n$ to a polynomial p in the unknowns corresponding to a_1, \ldots, a_n .
- If the proposed model is decomposable, then there is exactly one unknown for each $a_i \in A_i$, so the set of all unknowns is $Y = \bigcup_{i=1}^{n} A_i$.
- If the proposed model is not decomposable, then there may be more than one unknown for some a_i. In this case, the set of all unknowns is still designated Y.
- Define the relation \succeq_I on the set of polynomials corresponding to some $a_1 \cdots a_n$ such that when p corresponds to $a_1 \cdots a_n$ and q to $b_1 \cdots b_n$, we have $p \succeq_I q$ iff $a_1 \cdots a_n \succeq b_1 \cdots b_n$.



Measurement Inequalities Polynomial Structures Polynomial Structures

Polynomial Structures Representation Theorem: Big Picture

heorem 4

Interview of a set of playmonial inequalities in the unknowns Y has a solution if the corresponding relation χ_r on $\mathbb{R}[Y]$ can be extended to a weak order χ_B such that $(\mathbb{R}[Y], \chi_B)$ is an Archimedian weakly ordered ring (eq. χ_B induces an Archimedian ordered ring structure on $\mathbb{R}[Y] / _{\chi_B}$).

 We can find necessary conditions for this extension to exist similar to the necessary and sufficient conditions from the linear case.

 However, the necessary and sufficient conditions for the extension do not imply any easily testable consequences.

Polynomial Structures

Representation Theorem: Big Picture

Theorem 4

A set of polynomial inequalities in the unknowns Y has a solution iff the corresponding relation \succeq_I on $\mathbb{R}[Y]$ can be extended to a weak order \succeq_{II} such that $\langle \mathbb{R}[Y], \succeq_{II} \rangle$ is an Archimedian weakly ordered ring (i.e., \succeq_{II} induces an Archimedian ordered ring structure on $\mathbb{R}[Y] \nearrow_{II}$).

- We can find necessary conditions for this extension to exist similar to the necessary and sufficient conditions from the linear case.
- However, the necessary and sufficient conditions for the extension do not imply any easily testable consequences.



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 Measurement Inequalities

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 Polynomial Structures

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 Polynomial Structures

Polynomial Structures Representation Theorem: Necessary Condition

rollary to Theorem

If there exists an extension $\gtrsim' ol \gtrsim_I$ such that $\langle \mathbb{R}[Y], \gtrsim' \rangle$ is a weakly ordered ring, then \succ^* is irreflexive, where (\sim^*, \succ^*) is the minimal regular extension of \gtrsim_I .

Any binary relation on $\mathbb{R}[Y]$ has at least one regular extension (the universal extension) and a unique minimal regular extension • The universal extension is $\mathbb{R}[Y] \times \mathbb{R}[Y]$

Polynomial Structures

Representation Theorem: Necessary Conditions

Corollary to Theorem 5

If there exists an extension \succeq' of \succeq_I such that $\langle \mathbb{R}[Y], \succeq' \rangle$ is a weakly ordered ring, then \succ^* is irreflexive, where (\sim^*, \succ^*) is the minimal regular extension of \succeq_I .

Theorem 5

Any binary relation on $\mathbb{R}[Y]$ has at least one regular extension (the universal extension) and a unique minimal regular extension.

• The universal extension is $\mathbb{R}[Y] \times \mathbb{R}[Y]$



Polynomial Structures Representation Theorem: Necessary Condi

Regular Extension

 $\label{eq:approximation} \{ \begin{array}{l} \gamma_{2}, \gamma_{2} \} \mbox{is called a regular extension of <math display="inline">\gamma_{1}$ off $(p, p, q, q) \mbox{array on the fulf factoring to <math display="inline">q, q$ of $(p, p, q, q) \mbox{array on the fulf factoring to <math display="inline">q, q$ of $(p, q, q) \mbox{array on the fulf factoring to <math display="inline">q, q$ of $(p, q, q) \mbox{array on the fulf factoring to <math display="inline">q, q$ of $(p, q, q) \mbox{array on the fulf factoring to <math display="inline">q, q$ of $(p, q, q) \mbox{array on the fulf factoring factoring to <math display="inline">q, q$ of $(p, q) \mbox{array on the fulf factoring factoring$

Polynomial Structures

Representation Theorem: Necessary Conditions

Regular Extension

A pair of relations (\sim_{II}, \succ_{II}) is called a *regular extension* of \succeq_{I} iff

(a) $p \sim_{II} q$ whenever one of the following holds:

- (i) Extension: $p \sim_I q$
- (ii) Polynomial Identity: p = q
- (iii) Closure: There are p_1, p_2, q_1, q_2 with $p_1 \sim_{II} q_1, p_2 \sim_{II} q_2$ such that either $p = p_1 + p_2, q = q_1 + q_2$ or $p = p_1 p_2, q = q_1 q_2$.
- (b) $p \succ_{II} q$ whenever one of the following holds:
 - (i) Extension: $p \succ_I q$
 - (ii) Additive Closure: There are p_1, p_2, q_1, q_2 with $p_1 \succ_{II} q_1$,

 $p_2 \sim_{II} q_2$ such that $p = p_1 + p_2$ and $q = q_1 + q_2$.

(iii) Multiplicative Closure: There are p_1, q_1, r with either $p_1 \succ_{II} q_1$, $r \succ_{II} 0$ or $q_1 \succ_{II} p_1, 0 \succ_{II} r$ such that $p = p_1 r, q = q_1 r$.

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Polynomial Structures Necessary and Sufficient Conditions

orem 6

A set of polynomial inequalities in the unknowns of Y has a solution iff the corresponding relation \gtrsim_I on $\mathbb{R}[Y]$ has a regular extension (\sim_R,\succ_R) such that \gtrsim_R is Archimedean and \succ_R is non-universal.

Conjecture

There exists an extension \gtrsim' of \gtrsim_{ℓ} such that $(\mathbb{R}[Y], \gtrsim')$ is a weakly ordered ring iff \succ^* is irreflexive, where (\sim^*, \succ^*) is the minimal regular extension of \succeq_d .

- 1. (read slide)
- 2. There is a paper from the JOURNAL OF MATHEMATICAL PSYCHOLOGY 12, 99-113 (1975) by Marcel Richter that may or may not actually decide this conjecture, but at any rate gives an algebraic criterion for the solvability of arbitrary finite sets of polynomial inequalities.

Polynomial Structures

Necessary and Sufficient Conditions

Theorem 6

A set of polynomial inequalities in the unknowns of *Y* has a solution iff the corresponding relation \succeq_I on $\mathbb{R}[Y]$ has a regular extension (\sim_{II}, \succ_{II}) such that \succeq_{II} is Archimedean and \succ_{II} is non-universal.

Conjecture

There exists an extension \succeq' of \succeq_I such that $\langle \mathbb{R}[Y], \succeq' \rangle$ is a weakly ordered ring iff \succ^* is irreflexive, where (\sim^*, \succ^*) is the minimal regular extension of \succeq_I .



(transition)

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The Big Picture

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The Big Picture



The Big Picture

MY HOBBY: ABUSING DIMENSIONAL ANALYSIS







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Physical Laws

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Physical Laws

Examples

■ *F* = ma

■ *p* = *mv*

• $E_k = \frac{1}{2}mv^2$ • $P = IR^2$

• $F = G \frac{m_1 m_2}{r^2}$

POWER COMES GREAT CURRENT SQUARED TIMES RESISTANCE.

REMEMBER: WITH GREAT



OHM NEVER FORGOT HIS DYING UNCLE'S ADVICE.



Physical Laws What's no Special? Several units of measurement are expressible in terms of

others. = Taking (*Q*), temperature (Θ), mass (*M*), length (*L*), time duration (*T*), and angle (*A*) as primary, all other known physical attributes are expressible as monomial combinations of these

The Density dimensions of ML^{-3} = Frequency: dimensions of $T^{-1}A$ = Force: dimensions of MT^{-2} = Current: dimensions of QT^{-1} = Entropy: dimensions of QT^{-1}

= In fact, all the meaningful monomial combinations known are relatively simple: $Q^{\chi}\Theta^{\theta}M^{\mu}L^{\lambda}T^{\nu}A^{\mu}$ where $\chi, \theta, \mu, \lambda, \tau, \alpha$ are all small integers (between -4 and 4). Physical Laws What's so Special?

- Several units of measurement are expressible in terms of others.
- Taking charge (Q), temperature (Θ), mass (M), length (L), time duration (T), and angle (A) as primary, all other known physical attributes are expressible as monomial combinations of these.
 - \Box Density: dimensions of ML^{-3}
 - \Box Frequency: dimensions of $T^{-1}A$
 - □ Force: dimensions of MLT^{-2}
 - \Box Current: dimensions of QT^{-1}
 - □ Entropy: dimensions of $\Theta^{-1}ML^2T^{-2}$
- In fact, all the meaningful monomial combinations known are relatively simple: $Q^{\chi}\Theta^{\theta}M^{\mu}L^{\lambda}T^{\tau}A^{\alpha}$ where $\chi, \theta, \mu, \lambda, \tau, \alpha$ are all small integers (between -4 and 4).

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Dimensional Analysis

Physical Laws Wrat's so Special?

- Furthermore, there are some "dimensional constant," that relate various measurements. Some are system-dependent, others are truly constant for a food system of units:
 System-dependent gwaitational constant g (e.g., approx. 9.8m/s² for Earth)
 Violotity of light c, electron charge e, gas constant R, Planck's constant R, Mongdo's constant Ra,
- Certain measures such as momentum and kinstic energy are useful in many laws, but no laws seem to play a role in defining them. They are like the density of objects, not the density of materials (density independent of volume).
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Furthermore, most quantities of the form $m^i v^j$ aren't ter important.

Physical Laws What's so Special?

- Furthermore, there are some "dimensional constants" that relate various measurements. Some are system-dependent, others are truly constant for a fixed system of units:
 - □ System-dependent gravitational constant g (e.g., approx. $9.8m/s^2$ for Earth)
 - □ Velocity of light c, electron charge e, gas constant R, Planck's constant h, Avogadro's constant N_A
- Certain measures such as momentum and kinetic energy are useful in many laws, but no laws seem to play a role in defining them. They are like the density of objects, not the density of materials (density independent of volume).
- Furthermore, most quantities of the form $m^i v^j$ aren't terribly important.

C Dimensional Analysis Physical Laws Physical Laws Physical Laws The Big Questions

So what role are laws playing?
 Why are laws generally so simple?
 Why does the dimensional analysis heuristic work? (The only meaninghle quartonis (additions) are those where the sides (terms) have matching dimensions)

Physical Laws The Big Questions

- So what role are laws playing?
- Why are laws generally so simple?
- Why does the dimensional analysis heuristic work? (The only meaningful equations (additions) are those where the sides (terms) have matching dimensions)



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Outline

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The Algebra of Physical Quantities

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The Algebra of Physical Quantities

General Requireme

 Quantities with the same (extensively measurable) units combine additively.
 Ouantities with different dimensions combine multiplicatively.

- $\hfill \hfill \hfill$
- The existence of basic dimensions is analogous to the existence of a finite basis of that vector space.
 Numerical physical laws are formulated in terms of a very
- Numerical physical laws are formulated in terms of a very special class of functions on the space.

The Algebra of Physical Quantities

General Requirements

- Quantities with the same (extensively measurable) units combine additively.
- Quantities with different dimensions combine multiplicatively.
- The multiplicative structure resembles a finite-dimensional vectors space over Q.
- The existence of basic dimensions is analogous to the existence of a finite basis of that vector space.
- Numerical physical laws are formulated in terms of a very special class of functions on the space.

Dimensional Analysis

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—The Algebra of Physical Quantities

The Algebra of Physical Quantities

Structure of Physical Quantit

Suppose $A \leftarrow \mathbb{R}$ is a screenpty set $A^+ \subset A$ convergely, and $* A < A \rightarrow A$. The $A \in A^+ \subset A$ is the struct of physical quantization $\mathbb{R}(A \setminus \{0\}, s)$ is an abelian paper strends of $(\mathbb{R} \setminus \{0\}, s)$ and $1 + i \in associations and commutative.$ $21 <math display="inline">\mathbb{R}(A^- = \mathbb{R}^+)$. $\mathbb{R}_2 \to \mathbb{R}$ and 0 = 0. $\mathbb{R}_2 \to \mathbb{R}$ (there match panel of a = 0. $\mathbb{R}_2 \to \mathbb{R}$ (there match panel of a = 0. $\mathbb{R}_2 \to \mathbb{R}$ (there match panel a = 1 = a is in A^+ . $\mathbb{R}_2 \to \mathbb{R}$ (there match panel a = 1 = a is in A^+ . $\mathbb{R}_2 \to \mathbb{R}$ (there match panel a = 1 = a is a simple $A^{1,0}$. $\mathbb{R}_2 \to \mathbb{R}$ (there match panel a = 0. $\mathbb{R}_2 \to \mathbb{R}$ (there match panel a = 1 and a = 1 = a is in $A^{1,0}$. $\mathbb{R}_2 \to \mathbb{R}$ (there match panel a = 1). $\mathbb{R}_2 \to \mathbb{R}$ (there match panel a = 1). $\mathbb{R}_2 \to \mathbb{R}$ (there match panel a = 1). $\mathbb{R}_2 \to \mathbb{R}$ (there match panel a = 1). $\mathbb{R}_2 \to \mathbb{R}$ (there match panel a = 1). $\mathbb{R}_2 \to \mathbb{R}$ (there match panel a = 1). $\mathbb{R}_2 \to \mathbb{R}$ (there match panel a = 1). $\mathbb{R}_2 \to \mathbb{R}$ (there match panel a = 1). $\mathbb{R}_2 \to \mathbb{R}$ (there match panel a = 1). $\mathbb{R}_2 \to \mathbb{R}$ (there match panel a = 1). $\mathbb{R}_2 \to \mathbb{R}$ (there match panel a = 1). $\mathbb{R}_2 \to \mathbb{R}$ (there match panel a = 1). $\mathbb{R}_2 \to \mathbb{R}$ (there match panel a = 1). $\mathbb{R}_2 \to \mathbb{R}$ (there match panel a = 1). $\mathbb{R}_2 \to \mathbb{R}$ (there match panel a = 1). $\mathbb{R}_2 \to \mathbb{R}$ (there match panel a = 1). $\mathbb{R}_2 \to \mathbb{R}$ (there match panel a = 1). $\mathbb{R}_2 \to \mathbb{R}$ (there match panel a = 1). $\mathbb{R}_2 \to \mathbb{R}$ (there match panel a = 1). $\mathbb{R}_2 \to \mathbb{R}$ (there match panel a = 1). \mathbb{R} (there match panel a = 1). \mathbb{R} (there match panel a = 1). \mathbb{R} (there match panel a = 1) . \mathbb{R} (there match panel a = 1) . \mathbb{R} (there match panel a = 1) . \mathbb{R} (there match panel a = 1) . \mathbb{R} (

The Algebra of Physical Quantities Axiom System

Structure of Physical Quantities

Suppose $A \leftrightarrow \mathbb{R}$ is a nonempty set, $A^+ \subseteq A$ nonempty, and *: $A \times A \rightarrow A$. Then $\langle A, A^+, * \rangle$ is a *structure of physical quantities* iff $\langle A \setminus \{0\}, * \rangle$ is an abelian group extension of $\langle \mathbb{R} \setminus \{0\}, * \rangle$ and:

- 1. * is associative and commutative.
- 2. $\mathbb{R} \cap A^+ = \mathbb{R}^+$.
- 3. 1 * a = a and 0 * a = 0.
- 4. If $a \neq 0$, then exactly one of a and -1 * a is in A^+ .
- 5. If $x, y \in A^+$, then $x * y \in A^+$.
- 6. If $n \in \mathbb{Z}$, $n \neq 0$ and $x \in A^+$, there exists a unique $x^{1/n} \in A^+$ such that $(x^{1/n})^n = x$.
Dimensional Analysis Dimensional Analysis The Algebra of Ph The Algebra of

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$$\begin{split} & \text{if } xt = (\mathbb{R} \setminus \{0\} \times \mathbb{Q} \times \mathbb{Q}) \cup \{x\}, \\ & \text{ The transformation of the second second$$

The Algebra of Physical Quantities Example

• Let $A = (\mathbb{R} \setminus \{0\} \times \mathbb{Q} \times \mathbb{Q}) \cup \{z\}.$

- □ The element (α, q, r) can represent a quantity α with units $L^q M^r$. □ The element *z* represents 0.
- Let * on A be defined for non-zero operands as $(\alpha, q, r) * (\alpha', q', r') = (\alpha \alpha', q + q', r + r')$. Let z * a = a * z = z for all $a \in A$.

•
$$\mathbb{R} \hookrightarrow A$$
 by $\alpha \mapsto \begin{cases} (\alpha, 0, 0) & \alpha \neq 0 \\ z & \alpha = 0 \end{cases}$

• $A^+ = \{(\alpha, q, r) | \alpha \in \mathbb{R}^+\}$

•
$$(\alpha, q, r)^{-1} = (\alpha^{-1}, -q, -r)$$

• $(\alpha, q, r)^{1/n} = (\alpha^{1/n}, q/n, r/n)$ for $(\alpha, q, r) \in A^+$, $n \in \mathbb{Z}$.

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$$\begin{split} & \text{Let } (A,A^{*},*) \text{ be a structure of physical quantities.} \\ & \text{For } a_{f} \neq z \in A, \text{ define } [j] = \{\alpha * a | \alpha \in \mathbb{R}\}, [j^{*}] = [j] \cap A^{*}. \\ & \text{Th set } [A] = \{[a] | z| > a\} \text{ is a structure of equivalence classes over } A, \\ & \text{and each equivalence class can be thought of as a dimension.} \\ & \text{Three are well-chardenge operations } [j] \in [j] = [a] \in A], \\ & \text{Three are well-chardenge operations } [j] \in [j] = [a] \in A], \\ & \text{for a structure of a$$

Theorem 1 Suppose that $(A, A^+, *)$ is a structure of physical quantities. Then the set [A] under * and powers as defined above is a multiplicative vector space over \mathbb{Q} where $[1] = \mathbb{R}$ is the identity deterent and $[a]^{-1} = [a^{-1}]$ is the inverse of [a].

1. Recall that Theorem 1 was one of our desiderata.

The Algebra of Physical Quantities Dimension Space

- Let $\langle A, A^+, * \rangle$ be a structure of physical quantities.
- For $a \neq z \in A$, define $[a] = \{\alpha * a | \alpha \in \mathbb{R}\}$, $[a^+] = [a] \cap A^+$.
- The set [A] = {[a]|a ∈ A} is a set of equivalence classes over A, and each equivalence class can be thought of as a dimension.
- There are well-defined operations [a] * [b] = [a * b] and $[x]^{\rho} = [x^{\rho}] = [(x^1/j)^i]$ for $x \in A^+$, $\rho = i/j$, $i, j \in \mathbb{Z}$.

Theorem 1

Suppose that $\langle A, A^+, * \rangle$ is a structure of physical quantities. Then the set [*A*] under * and powers as defined above is a multiplicative vector space over \mathbb{Q} where $[1] = \mathbb{R}$ is the identity element and $[a]^{-1} = [a^{-1}]$ is the inverse of [a].

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heorem 2

Interface z_i , (A,A^+,v) is a structure of physical quantities. Then the elements $z_1,\ldots,z_k\in A^+$ span A iff for every $a\in A$ there exist $a\in \mathbb{R}$ and $p_i,\ldots,p_k\in Q$ such that $a=\alpha\circ x_1^{A_1}\cdots x_k^{A_k}$. They are independent iff $a_1^{21}\circ\cdots\circ a_k^{A_k}\in\mathbb{R}$ implies that $r_1=0$ for all i. If they are independent, they are a basis for [A] and the p_i depend only on [a].

 The dimensions that are elements of a basis for [A] can be thought of as basic/fundamental dimensions.

We can also introduce a formal addition within a dimension: □ Suppose a, b ∈ [c] where a = α × c, b = β × c, and α, β ∈ ℝ. Then define a ⊕ b = (α + β) × c

- 1. These are two more of our desiderata.
- 2. The formal addition agrees with the extensive concatenation operation if the dimension is extensively measurable.

The Algebra of Physical Quantities

Theorem 2

Suppose that $\langle A, A^+, * \rangle$ is a structure of physical quantities. Then the elements $a_1, \ldots, a_n \in A^+$ span A iff for every $a \in A$ there exist $\alpha \in \mathbb{R}$ and $\rho_1, \ldots, \rho_n \in \mathbb{Q}$ such that $a = \alpha * a_1^{\rho_1} * \cdots * a_n^{\rho_n}$. They are independent iff $a_1^{\gamma_1} * \cdots * a_n^{\gamma_n} \in \mathbb{R}$ implies that $\gamma_i = 0$ for all i. If they are independent, they are a basis for [A] and the ρ_i depend only on [a].

- The dimensions that are elements of a basis for [A] can be thought of as basic/fundamental dimensions.
- We can also introduce a formal addition within a dimension:
 - □ Suppose $a, b \in [c]$ where $a = \alpha * c$, $b = \beta * c$, and $\alpha, \beta \in \mathbb{R}$. Then define $a \oplus b = (\alpha + \beta) * c$

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The Algebra of Physical Quantities Functional Ferm Restrictions

• Let $(A, A^+, *)$ be a structure of physical quantities. • Let $P = [a] \cap A^+$ be a typical positive dimension. • Physical laws have the following form: • A function $F_1 \times \cdots \times P_n$ where $n \geq 2$. • A condition $f(n_1, \dots, n_n) = 0$ on the physically realizable values of $x \in P_n$.

 The functional form of laws are (usually) restricted to be dimensionally invariant (homogeneous). This means that the function should be invariant under changes of units between coherent systems.

1. In the condition on realizable values, the x_i are usually treated as real numbers, but in fact, they involve the specification of both the dimension and the unit of measurement of which the numerical dimensionless ration x_i is given.

The Algebra of Physical Quantities Functional Form Restrictions

- Let $\langle A, A^+, * \rangle$ be a structure of physical quantities.
- Let $P = [a] \cap A^+$ be a typical positive dimension.
- Physical laws have the following form:
 - $\Box \text{ A function } f: P_1 \times \cdots \times P_s \to \mathbb{R}, \text{ where } s \geq 2.$
 - □ A condition $f(x_1, ..., x_s) = 0$ on the physically realizable values of $x_i \in P_i$.
- The functional form of laws are (usually) restricted to be dimensionally invariant (homogeneous). This means that the function should be invariant under changes of units between coherent systems.



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The Pi Theorem and Dimensional Analysis

The Pi Theorem and Dimensional Analysis

The Pi Theorem and Dimensional Analysis

imilarity

Suppose that $(A, A^+, *)$ is a structure of physical quantities. A function $\phi : A \rightarrow A$ is a similarity iff it is an automorphism of A that preserves dimensions, maps A^+ into itself, and fixes $\alpha \in \mathbb{R}$.

Theorem 3 Suppose that a structure of physical quantities $(A, A^+, *)$ is of finite dimension and that $\{a_1, ..., a_n\}$ is a basis. If ϕ is a similarity

one A, then there are numbers $\phi_i > 0$ such that $\phi(x_i) = \phi_i + a$ and so $\phi_i > 0$ such that $\phi(x_i) = \phi_i + a$ and so $\phi_i = 0$ ($\phi_1^{i_1} \dots \phi_n^{i_n}$) a, where $a = a + a_i^{i_1} + \dots + a_n^{i_n}$. Conversely, for any $\phi_i > 0$, the function $\phi(a) = (\phi_1^{i_1} \dots \phi_n^{i_n}) + a$ is a similarity.

The Pi Theorem and Dimensional Analysis Similarities

Similarity

Suppose that $\langle A, A^+, * \rangle$ is a structure of physical quantities. A function $\phi : A \to A$ is a *similarity* iff it is an automorphism of A that preserves dimensions, maps A^+ into itself, and fixes $\alpha \in \mathbb{R}$.

Theorem 3

Suppose that a structure of physical quantities $\langle A, A^+, * \rangle$ is of finite dimension and that $\{a_1, \ldots, a_n\}$ is a basis. If ϕ is a similarity on A, then there are numbers $\phi_i > 0$ such that $\phi(a_i) = \phi_i * a$ and so $\phi(a) = (\phi_1^{\rho_1} \cdots \phi_n^{\rho_n}) * a$, where $a = \alpha * a_1^{\rho_1} * \cdots * a_n^{\rho_n}$. Conversely, for any $\phi_i > 0$, the function $\phi(a) = (\phi_1^{\rho_1} \cdots \phi_n^{\rho_n}) * a$ is a similarity.

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The Pi Theorem and Dimensional Analysis

The Pi Theorem and Dimensional Analysis

Dimensional Invariance Suppose that $(A, A^+, *)$ is a structure of physical quantities and that P_1 are positive dimensions. A function $f : \prod_{i=1}^{n} P_i \to \mathbb{R}$ is dimensionally invariant iff for all similarities ϕ on A, $(P_{i_1,...,i_k}) = 0$ if $(\phi(x_{i_1},...,\phi(x_k))) = 0$.

The Pi Theorem and Dimensional Analysis

Dimensional Invariance

Dimensional Invariance

Suppose that $\langle A, A^+, * \rangle$ is a structure of physical quantities and that P_i are positive dimensions. A function $f : \prod_{i=1}^n P_i \to \mathbb{R}$ is *dimensionally invariant* iff for all similarities ϕ on A, $f(x_1, \ldots, x_n) = 0$ iff $f(\phi(x_1), \ldots, \phi(x_n)) = 0$.

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The Pi Theorem and Dimensional Analysis

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orem 4

Suppose that $(A,A^+,*)$ is a finite-dimensional structure of physical quantities, that $P_i, = 1, \ldots, s$ are positive dimensions of the structure that are indexed so that the first t < s form a maximal independent subset of the subspace spanned by all s of them, and that $f:\prod_i P_i \to \mathbb{R}$ is a dimensionality invariant

function. Then there exist a function $F\colon \mathbb{R}^{t-r}\to\mathbb{R}$ and $\rho_{ij}\in\mathbb{Q}$ for $i=r+1,\ldots,s,j=1,\ldots,s,r$ such that for all $x_i\in P_i,$ $x_{irre}=x+x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_i^{-rre},s=x_$

1. To understand this, it helps to have a bit of an "example".

The Pi Theorem and Dimensional Analysis

Theorem 4

Suppose that $\langle A, A^+, * \rangle$ is a finite-dimensional structure of physical quantities, that P_i , i = 1, ..., s are positive dimensions of the structure that are indexed so that the first r < s form a maximal independent subset of the subspace spanned by all *s* of them, and that $f: \prod P_i \to \mathbb{R}$ is a dimensionally invariant function. Then there exist a function $F : \mathbb{R}^{s-r} \to \mathbb{R}$ and $\rho_{ii} \in \mathbb{Q}$ for $i = r + 1, \ldots, s, i = 1, \ldots, r$ such that for all $x_i \in P_i$, $\pi_{i-r} = x_i * x_1^{-\rho_{i1}} * \cdots * x_r^{-\rho_{ir}}$, for $i = r + 1, \dots, s$, are real numbers (dimensionless), and $f(x_1, \ldots, x_s) = 0$ iff $F(\pi_1, \ldots, \pi_{s-r}) = 0$. Conversely, any function of the π 's as above is dimensionally invariant.

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Dimensional Analysis

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- The Pi Theorem and Dimensional Analysis
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The Pi Theorem and Dimensional Analysis The Pi Theorem: "Example"

* A physical law usually represents a dependent worklike in terms of several independent near $_{-}=q(c_1,\ldots,c_{-})$. * Using the Pl Theorem, we can solitch this is a dimensionless term $\pi_{+}=q(c_1,\ldots,c_{+})$. * We can also go backwards and express this as: $\chi_{-}=\chi_{-}^{-1}=\chi_{-}^{-1}=\chi_{-}^{-1}=(c_{+}-c_{+})$. * The function G give a propertional constant relating χ_{+} a moremial of the independent dimension $\chi_{+}\ldots,\chi_{+}$ for

The Pi Theorem and Dimensional Analysis The Pi Theorem: "Example"

- A physical law usually represents a dependent variable in terms of several independent ones: $x_s = g(x_1, \dots, x_{s-1})$.
- Using the Pi Theorem, we can switch this to a dimensionless form: $\pi_{s-r} = G(\pi_1, \dots, \pi_{s-r-1}).$
- We can also go backwards and express this as: $x_s = x_1^{\rho_{s1}} * \cdots * x_r^{\rho_{sr}} * G(\pi_1, \dots, \pi_{s-r-1}).$
 - □ The function *G* gives a proportional constant relating x_s to a monomial of the independent dimensions x_1, \ldots, x_r .



(transition)

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Examples of Dimensional Analysis

What is the period of oscillation, t, of a simple pendulum?

The behavior of a simple pendulum has the five parameters: = t for time = l for the length of the pendulum = α for the angle from vertical = m for the mass of the pendulum = g for the gravitational acceleration

Examples of Dimensional Analysis A Simple Pendulum

What is the period of oscillation, *t*, of a simple pendulum?

The behavior of a simple pendulum has the five parameters:

- *t* for time
- I for the length of the pendulum
- α for the angle from vertical
- *m* for the mass of the pendulum
- **g** for the gravitational acceleration

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we can wri	te down the	foll	owi	na te	able:		
-		PI	Physical quantities				
	Dimensions	t	1	199	8	ά	
	L	0	1	0	1	0	
	м	0	0	1	0	0	
	T	1	0	0	-2	0	

Examples of Dimensional Analysis A Simple Pendulum

	Ρŀ	nysi	cal c	luant	ities
Dimensions	t	1	т	g	α
L	0	1	0	1	0
М	0	0	1	0	0
Т	1	0	0	-2	0



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 - Examples of Dimensional Analysis

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A Simple People	lum		

Since there are three dimensions and five parameters, by the Pi Theorem, there must be 2 dimensionless parameters π_1 and π_2 . Clearly one of these is $\pi_2=\alpha$. We can use standard linear algebra to find the other.

Examples of Dimensional Analysis A Simple Pendulum

Thus, we can write down the following table:

	Pŀ	nysi	cal q	luant	ities
Dimensions	t	1	т	g	α
L	0	1	0	1	0
М	0	0	1	0	0
Т	1	0	0	-2	0

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	Р	Physical guarr				
Dimens	ions t	7	199	8	ά	
L	0	1	0	1	0	
M	0	0	1	0	0	
T	1	0	0	-2	0	

Exa

$$\begin{split} L &: 0\rho_t + 1\rho_l + 0\rho_m + 1\rho_g = 0 \\ M &: 0\rho_t + 0\rho_l + 1\rho_m + 0\rho_g = 0 \\ \mathcal{T} &: 1\rho_t + 0\rho_l + 0\rho_m - 2\rho_g = 0 \end{split}$$

Examples of Dimensional Analysis A Simple Pendulum

	Pł	nysi	cal c	luant	ities
Dimensions	t	1	т	g	α
L	0	1	0	1	0
М	0	0	1	0	0
Т	1	0	0	-2	0

$$L: 0\rho_t + 1\rho_l + 0\rho_m + 1\rho_g = 0$$

$$M: 0\rho_t + 0\rho_l + 1\rho_m + 0\rho_g = 0$$

$$T: 1\rho_t + 0\rho_l + 0\rho_m - 2\rho_g = 0$$

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Examples of Dimensional Analysis A Simple Pendulum

	Pŀ	nysi	cal c	luant	ities
Dimensions	t	1	т	g	α
L	0	1	0	1	0
М	0	0	1	0	0
Т	1	0	0	-2	0

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} \rho_t \\ \rho_l \\ \rho_m \\ \rho_g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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		_	PI	nusi	cal c	wan	tities
Di	nensie	ans.	t	1	199	×	ά
_	L		0	1	0	1	0
	М		0	0	1	0	0
_	T		-1	0	0	-2	0
0 0 1	1 0 0 1 0 0	1	2	~ [1 0 0 1 0 0	0 0 1	-2 1 0

Examples of Dimensional Analysis A Simple Pendulum

	Ρŀ	nysi	cal c	luant	ities
Dimensions	t	1	т	g	α
L	0	1	0	1	0
М	0	0	1	0	0
Т	1	0	0	-2	0

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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	_		DI		ol z		(*2.44
	Dir	nensions	÷	1	m	¥.	a
	_	L	0	1	0	1	0
		М	0	0	1	0	0
		T	1	0	0	-2	0

Examples of Dimensional Analysis A Simple Pendulum

Thus, we can write down the following table:

	Pŀ	nysi	cal c	luant	ities
Dimensions	t	1	т	g	α
L	0	1	0	1	0
М	0	0	1	0	0
Т	1	0	0	-2	0

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We can choose one ρ arbitrarily. Since *t* is the dependent variable, it is customary to set $\rho_t = 1$.

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		DI		ent e		112.00
	Dimensions	-	1	m	guant #	a
-	L	0	1	0	1	0
	м	0	0	1	0	0
	T	1	0	0	-2	0
- [1 0 0 -: 0 1 0 1	2 C	1 lear	0 ly p m ti	0 m = w fu	-2 0, an st ro	d it w th

Examples of Dimensional Analysis A Simple Pendulum

Thus, we can write down the following table:

-							
	Physical quantities						
Dimensions	t	Ι	т	g	α		
L	0	1	0	1	0		
М	0	0	1	0	0		
Т	1	0	0	-2	0		

 $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ Clearly $\rho_m = 0$, and it is easy to see from the first row that $\rho_g = \frac{1}{2}$. Finally, then, $\rho_l = -\frac{1}{2}$.

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		_	_			
		P	nysi	cal c		ities
Dir	winsions	t	1	199	8	ά
	L	0	1	0	1	0
	M	0	0	1	0	0
	T	1	0	0	-2	0
$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	T Sin the	'her ce s	efor NO C	e, w san v = Φ	e hav vrite (a) ($e_{\pi_2} = \frac{\pi_2}{\pi_2} = \frac{\pi_2}{4}$

Examples of Dimensional Analysis A Simple Pendulum

	Physical quantities						
Dimensions	t	1	т	g	α		
L	0	1	0	1	0		
М	0	0	1	0	0		
Т	1	0	0	-2	0		

Therefore, we have
$$\pi_2 = t \left(\frac{g}{l}\right)^{1/2}$$

Since we can write $\pi_2 = G(\pi_1)$, we
then get $t = \Phi(\alpha) \left(\frac{l}{g}\right)^{1/2}$.

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Possible Errors: A Simple Pendulum

- Suppose gravitational mass and inertial mass were assumed equivalent. Then mass would have dimension $L^3 T^{-2}$. (274)
- If we walked through the simple pendulum example again, we would start with just the table:

Dimensions	t	1	т	g	α
L	0	1	3	1	0
Т	1	0	-2	-2	0

• We would then arrive at $\pi_1 = \alpha$, $\pi_2 = I\left(\frac{g}{m}\right)^{1/2}$, and $\pi_3 = t\left(\frac{g}{l}\right)^{1/2}$.

• Therefore, we would arrive at $t = \Phi\left(l(g/m)^{1/2}, \alpha\right) \left(\frac{l}{g}\right)^{1/2}$, which is not technically wrong, but is misleading.

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Examples of Dimensional Analysis Possible Errors: Ballistics

 In some situations, we may try to include more dimension: than recessary, such as with the ballicity as mapple on 475-6. This generally last to a more complete solution.
 Other times, redundant bases and superflaxes constants may be included. This generally result in the inclusion of universal constants that can be chosen to be convenient values, reducing the solution to a non-redundant case.

Examples of Dimensional Analysis Possible Errors: Ballistics

- In some situations, we may try to include more dimensions than necessary, such as with the ballistics example on 475-6. This generally leads to a more complete solution.
- Other times, redundant bases and superfluous constants may be included. This generally results in the inclusion of universal constants that can be chosen to be convenient values, reducing the solution to a non-redundant case.

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Examples of Dimensional Analysis Obtaining Exact Solutions

= We can also use dimensional analysis to help obtain exact solutions to some partial differential equations by reducing the space of possible solutions. = Example: Propagation of vorticity is given by $\frac{\partial \Omega}{\partial t} = \nu \left(\frac{\partial^2 \Omega}{\partial r^2} + \frac{1}{r} \frac{\partial \Omega}{\partial r} \right)$

 $\partial t = - (\partial r^2 + r \partial r)$ = Here, Ω is the angular velocity of a viscous fluid, r is the radial distance, t is time, and $\nu = \mu/d$ is the kinematic viscosity. = Suppose we want to solve for $\Omega(r, t)$ subject to the initial confittion that the circulation around a circle of radius. R at the

origin is a constant, i.e.: $\Gamma = 4\pi \int_{-\infty}^{\infty} r\Omega(r, 0) dr$

Examples of Dimensional Analysis Obtaining Exact Solutions

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- Here, Ω is the angular velocity of a viscous fluid, r is the radial distance, t is time, and $\nu = \mu/d$ is the kinematic viscosity.
- Suppose we want to solve for $\Omega(r, t)$ subject to the initial condition that the circulation around a circle of radius R at the origin is a constant, i.e.: $\Gamma = 4\pi \int_{0}^{R} r\Omega(r, 0) dr$

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$$\begin{split} & \mbox{Examples of Dimensional Analysis} \\ & \mbox{Dimensional Analysis} \\ & \mbox{Dimensional Constant Co$$

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Obtaining Exact Solutions Continued

Given this problem description, we can set up a dimensional analysis for $\Omega(\Gamma, \nu, r, t)$:

Dimensions	Ω	Г	ν	r	t
L	0	2	2	1	0
М	0	0	0	0	0
Т	-1	-1	-1	0	1

By the Pi Theorem, there are two dimensionless parameters, namely $\pi_1 = r^2 \nu^{-1} t^{-1}$ and $\pi_2 = \Omega \nu t \Gamma^{-1}$, so we have, where $\xi = r^2 / \nu t$:

$$\Omega = (\Gamma/\nu t) \Phi(\xi)$$

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Examples of Dimensional Analysis Obtaining Exact Solutions Continued

Substituting $\Omega = (\Gamma/\nu t)\Phi(\xi)$ into $\frac{\partial\Omega}{\partial t} = \nu \left(\frac{\partial^2\Omega}{\partial t^2} + \frac{1}{2}\frac{\partial\Omega}{\partial t}\right)$ and simplifying, we get:

 $\frac{d}{dt}\left[\xi\Phi(\xi) + 4\xi\frac{d\Phi(\xi)}{d\xi}\right] = 0$

Therefore, it is clear that we must have $\xi \Phi(\xi) + 4\xi \frac{d\Phi(\xi)}{d\xi} = C$

 $\zeta + (\zeta) + 4\zeta - \frac{d\zeta}{d\xi} = c$

Assuming $\Phi(0)$ and $\frac{d\Phi(0)}{d\xi}$ are finite, setting $\xi=0$ shows that C=0.

Examples of Dimensional Analysis

Obtaining Exact Solutions Continued

Substituting $\Omega = (\Gamma/\nu t)\Phi(\xi)$ into $\frac{\partial\Omega}{\partial t} = \nu \left(\frac{\partial^2\Omega}{\partial r^2} + \frac{1}{r}\frac{\partial\Omega}{\partial r}\right)$ and simplifying, we get:

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Examples of Dimensional Analysis Obtaining Exact Solutions Continued

So, we can rewrite things as: $\frac{d\Phi(\xi)}{d\xi} = -\frac{1}{s}\Phi(\xi)$

From this it clearly follows that we must have $\Phi(\xi) = Ae^{-\xi/4}$, for some constant A.

Substituting this back into the expression for $\Omega,$ we get $\Omega(r,t)=(\Gamma A/\nu t)e^{-r^2/4\omega t}.$

Putting that back into the initial condition to solve for $A = \frac{1}{Rt}$, we arrive at the solution $\Omega(r, t) = (\Gamma/8\pi\nu t)e^{-t^2/4\nu t}$.

Examples of Dimensional Analysis

Obtaining Exact Solutions Continued

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$$rac{d\Phi(\xi)}{d\xi} = -rac{1}{4}\Phi(\xi)$$

From this it clearly follows that we must have $\Phi(\xi) = Ae^{-\xi/4}$, for some constant *A*.

Substituting this back into the expression for Ω , we get $\Omega(r, t) = (\Gamma A/\nu t)e^{-r^2/4\nu t}$.

Putting that back into the initial condition to solve for $A = \frac{1}{8\pi}$, we arrive at the solution $\Omega(r, t) = (\Gamma/8\pi\nu t)e^{-r^2/4\nu t}$.



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Consistency of Derived Measures

Embedding into a Structure of Physical Quantities Why are Numerical Laws Dimensionally Invariant?

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Consistency of Derived Measures

- The algebra of physical quantities gave us a way to describe how the ratio scale measures of various physical quantities combine, but it did not discuss the consistency of various measures obtained by other theories (e.g. extensive and corpint measurement).
- It is generally acknowledged that there are quantities that must be measured indirectly in terms of other extensive measures, and this is only possible because various physical laws are true.
- Furthermore, we want to show how extensive and conjoint measures can be embedded as a substructure of the theory we developed earlier.

Consistency of Derived Measures

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Consistency of Derived Measures

Consistency of Derived Measures

Consider a conjoint multiplicative scale a – bc where a and b can also be extensively measured. We would like some conditions to ensure consistency in the measurements.

Law of Similitud

Suppose that $(A_1 \times A_2 \gtrsim)$ is an additive complex structure and that $(A_2 \times A_2 \lesssim)$ and $(A_2 \lesssim)$, arc extensive structures. A (qualitative) fore of similined with exponents m and n, where $m \in \mathbb{Z}^n$ holds if then of the following is valid for all $a < A_n$ all $a \in A_n$ and $all <math>i \in \mathbb{Z}^n$, where the concatenations exist: (i) $[\sum_{i=1}^n \sum_{j=1}^n \sum_{i=1}^n a_i \sum_{j=1}^n \sum_{i=1}^n a_i d \sum_{i=1}^$

1. \succsim^* is one of \succsim or $\precsim,$ and similarly for the second structure.

Consistency of Derived Measures Laws of Similitude

Consider a conjoint multiplicative scale a = bc where a and b can also be extensively measured. We would like some conditions to ensure consistency in the measurements.

Law of Similitude

Suppose that $\langle A_1 \times A_2, \succeq \rangle$ is an additive conjoint structure and that $\langle A_1 \times A_2, \succeq^*, \circ \rangle$ and $\langle A_1, \succeq^*_1, \circ_1 \rangle$ are extensive structures. A (qualitative) *law of similitude with exponents m and n*, where $m, n \in \mathbb{Z}^+$ holds iff one of the following is valid for all $a \in A_1$, all $u \in A_2$, and all $i \in \mathbb{Z}^+$, where the concatenations exist:

(i) $\succeq^* = \succeq, \succeq^*_1 = \succeq_1 \text{ or } \succeq^* = \preceq, \succeq^*_1 = \preceq_1 \text{ and } i^m(a, u) \sim (i^n a, u)$ (ii) $\succeq^* = \succeq, \succeq^*_1 = \preceq_1 \text{ or } \succeq^* = \preceq, \succeq^*_1 = \succeq_1 \text{ and } (a, u) \sim i^m(i^n a, u)$

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 Consistency of Derived Measures
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Consistency of Derived Measures

Theorem 5

(motion 3) Suppose $(A_1 \times A_2, \sum)$ is a flat conjoint structure that has an additive representation $\log \alpha_1 + \log \alpha_2, |A_1 \times A_2 \gtrsim -\alpha_3$ and $A_1 \times A_2 \times \alpha_3$ and $A_2 \times \alpha_3$ representations with messential maximiz, α and α_1 are, respectively, additive extension scales, and that the range α_2 includes Q^* . If a law of similated with exponents m and n holds, then there are constants $\alpha_1 \times \alpha_3$, α_1 , and γ_1 such that:

 $\begin{array}{l} (i) \ \psi_1\psi_2=\gamma\phi^\alpha \ \text{and} \ \psi_1=\gamma_1\phi_1^{\alpha_1}\\ (i) \ \alpha>0 \ or<0 \ \text{according} \ as \gtrsim^+=\succsim \text{or} \ \precsim \ \text{and} \ \alpha_1>0 \ \text{or} <0 \\ \text{according} \ as \succeq^+=\varsigma_1 \ \text{or} \ \preccurlyeq_1 \end{array}$

1. A flat structure is one such that for every $a, b \in A_1$ there are $u, v \in A_2$ such that $(a, u) \sim (b, v)$.

Consistency of Derived Measures

Theorem 5

Suppose $\langle A_1 \times A_2, \succeq \rangle$ is a flat conjoint structure that has an additive representation $\log \psi_1 + \log \psi_2$; $\langle A_1 \times A_2, \succeq^*, \circ \rangle$ and $\langle A_1, \succeq_1^*, \circ_1 \rangle$ are closed extensive structures with no essential maxima; ϕ and ϕ_1 are, respectively, additive extensive scales; and that the range of ϕ_1 includes \mathbb{Q}^+ . If a law of similitude with exponents *m* and *n* holds, then there are constants α, γ, α_1 , and γ_1 such that:

(i)
$$\psi_1\psi_2 = \gamma\phi^{\alpha}$$
 and $\psi_1 = \gamma_1\phi_1^{\alpha_1}$
(ii) $\alpha > 0$ or < 0 according as $\gtrsim^* = \succeq$ or \preceq and $\alpha_1 > 0$ or < 0
according as $\succeq^*_1 = \succeq_1$ or \preceq_1
(iii) $|\alpha/\alpha_1| = n/m$
(iv) $|\alpha/\alpha_1| = n/m$

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Dimensional Analysis

Consistency of Derived Measures

-Consistency of Derived Measures

Consistency of Derived Measures

Now consider a conjoint multiplicative scale a = bc where b and c can also be extensively measured. We would again like some conditions to ensure consistency in the measurements.

Law of Exchange

Suppose $(A_1 \times A_2, \succeq)$ is an additive couplint structure and $(A_k, \succeq_{k-1}^{+}, \diamond_k)$, lend, Z, are extensive structures. A (qualitative) low of exchange with exponents m and n, where $m, n \in \mathbb{Z}^+$, holds iff once of the following is valid for all $a \in A_k$, all $u \in A_k$, and all $i \in \mathbb{Z}^+$, where the concatentations exist:

 $\begin{array}{l} \text{(i)} \hspace{0.1cm} \gtrsim_{1}^{a} = \hspace{-0.1cm} \gtrsim_{1}^{a} \hspace{-0.1cm} \gtrsim_{2}^{a} \hspace{-0.1cm} = \hspace{-0.1cm} \gtrsim_{1}^{a} \hspace{-0.1cm} \simeq_{1}^{a} \hspace{-0.1cm} \simeq_{2}^{a} \hspace{-0.1cm} \simeq_{2}^{a} \hspace{-0.1cm} \text{and} \hspace{0.1cm} (i^{m}a,u) \sim (a,i^{a}u) \\ \text{(ii)} \hspace{0.1cm} \succeq_{1}^{a} \hspace{-0.1cm} \simeq_{1}^{a} \hspace{-0.1cm} \simeq_{2}^{a} \hspace{-0.1cm} \simeq_{2}^{a}$

Consistency of Derived Measures Laws of Exchange

Now consider a conjoint multiplicative scale a = bc where b and c can also be extensively measured. We would again like some conditions to ensure consistency in the measurements.

Law of Exchange

Suppose $\langle A_1 \times A_2, \succeq \rangle$ is an additive conjoint structure and $\langle A_k, \succeq_k^*, \circ_k \rangle$, k=1,2, are extensive structures. A (qualitative) *law of exchange with exponents m and n*, where $m, n \in \mathbb{Z}^+$, holds iff once of the following is valid for all $a \in A_1$, all $u \in A_2$, and all $i \in \mathbb{Z}^+$, where the concatenations exist:

(i) $\gtrsim_1^* = \gtrsim_1$, $\gtrsim_2^* = \gtrsim_2$ or $\gtrsim_1^* = \preceq_1$, $\gtrsim_2^* = \preceq_2$ and $(i^m a, u) \sim (a, i^n u)$ (ii) $\gtrsim_1^* = \gtrsim_1$, $\gtrsim_2^* = \preceq_2$ or $\gtrsim_1^* = \preceq_1$, $\gtrsim_2^* = \gtrsim_2$ and $(a, u) \sim (i^m a, i^n u)$

Dimensional Analysis

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Consistency of Derived Measures

-Consistency of Derived Measures

Consistency of Derived Measures

Theorem

Increment 0 Suppose $(k_1 \times k_2, \sum)$ is a conjoint structure that has an additive representation $\log v_1 + \log v_2, (k_1, \sum_{i=1}^{2}, v_i), k = 1, 2,$ are closed positive extensive structures with one escential mains and additive representations v_i . It a law of exchange with exponents m and a holds, then there are constants α_k , and γ_k , k = 1, 2, such that: $(k) = v_k = \gamma_k \sigma_k^{-k}$

(i) $\forall k = |k \forall k|$ (ii) $\alpha_k > 0 \text{ or } < 0 \text{ according as } \gtrsim_k^* = \succeq_k \text{ or } \precsim_k$ (iii) $|\alpha_k / \alpha_2| = n/m$

Consistency of Derived Measures

Theorem 6

Suppose $\langle A_1 \times A_2, \succeq \rangle$ is a conjoint structure that has an additive representation log $\psi_1 + \log \psi_2$; $\langle A_k, \succeq_k^*, \circ_k \rangle$, k = 1, 2, are closed positive extensive structures with no essential maxima and additive representations ϕ_k . If a law of exchange with exponents *m* and *n* holds, then there are constants α_k , and γ_k , k = 1, 2, such that:

(i)
$$\psi_k = \gamma_k \phi_k^{\alpha_k}$$

(ii) $\alpha_k > 0$ or < 0 according as $\succeq_k^* = \succeq_k$ or \preccurlyeq_k
(iii) $|\alpha_1/\alpha_2| = n/m$

Consistency of Derived Measures

Consistency of Derived Measures Similitude and Exchange Compatibility

How compatible are the laws of similitude and exchange as given? Are more accumptions needed? Consider the case of conjust measurement where a — bc and a, b, c all have extension measurements. Two laws of similitude and one has of exchange could possible hold similarmoscill the first density is to density and the set of the s

can see that a compatible representation can be of the form $\phi^{eq} = \phi_1^{eq} \phi_2^{bp}$ or $\phi^{eq} = \phi_1^{ep} \phi_2^{ep}$. Similar conditions can be derived for cases larger than 2 dimensions

1. Skipping difference structures

Consistency of Derived Measures Similitude and Exchange Compatibility

How compatible are the laws of similitude and exchange as given? Are more assumptions needed?

• Consider the case of conjoint measurement where a = bc and a, b, c all have extensive measurements. Two laws of similitude and one law of exchange could possibly hold simultaneously. In this case, any two of the three possibly laws determine what the third must be for a representation of the form $\phi(a, u)^{\alpha} = \phi_1(a)^{\alpha_1}\phi_2(u)^{\alpha_2}$ to hold. With some manipulation, we

 $\phi(a, u)^{\alpha} = \phi_1(a)^{\alpha_1}\phi_2(u)^{\alpha_2}$ to hold. With some manipulation, we can see that a compatible representation can be of the form $\phi^{nq} = \phi_1^{mq}\phi_2^{np}$ or $\phi^{nq} = \phi_1^{np}\phi_2^{mp}$.

Similar conditions can be derived for cases larger than 2 dimensions.



Outline
Measurement Inequalities
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Enhedding into a Structure of Physical Quantities

(transition)

Outline

Measurement Inequalities

Solvability Finite Linear Structures Polynomial Structures

Dimensional Analysis

Physical Laws The Algebra of Physical Quantities The Pi Theorem and Dimensional Analysis Examples of Dimensional Analysis Consistency of Derived Measures Embedding into a Structure of Physical Quantities Why are Numerical Laws Dimensionally Invariant?

Dimensional Analysis

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-Embedding into a Structure of Physical Quantities

-Embedding into a Structure of Physical Quantities

Embedding into a Structure of Physical Quantities

= Let \mathscr{A} be a collection of physical attributes, represented by structures $\langle A, \gtrsim \rangle$, and let $\mathscr{E} \subset \mathscr{A}$ be a set of extensively measurable attributes, represented by structures $\langle A, \gtrsim, \circ \rangle$.

Embedding into a Structure of Physical Quantities

• Let \mathscr{A} be a collection of physical attributes, represented by structures $\langle A, \succeq \rangle$, and let $\mathscr{E} \subset \mathscr{A}$ be a set of extensively measurable attributes, represented by structures $\langle A, \succeq, \circ \rangle$.

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 - Embedding into a Structure of Physical Quantities

Embedding into a Structure of Physical Quantities

Axiomatize at and δ^{*} as follows:
 The set δ^{*} is nonempty and at is finite.
 If (A_i, z_i) ∈ δ_i it is an entempty structure with an additive representation whose range includes Q^{*}.
 If (A_i) ≥ q^{*} due it is part of a compart structure in the sense that entempt.
 (A = A_i × A₂, (A_i × A₂, z_i) is a symmetric conjust structure with a

(i) there is a summetric conjust structure $\langle A_1^i \times A_2^i, \sum_{i=1}^{n} \rangle \in a^i$ with a multiplicative representation such that $A_2^i = A_i \gtrsim_{k=1}^{i} = \gtrsim_{i=1}^{n}$ and $\langle A_2^i, \gtrsim_{k} \rangle \in a^i$.

Embedding into a Structure of Physical Quantities

- Axiomatize \mathscr{A} and \mathscr{E} as follows:
 - 1. The set $\mathscr E$ is nonempty and $\mathscr A$ is finite.
 - 2. If $\langle A, \succeq, \circ \rangle \in \mathscr{E}$, it is an extensive structure with an additive representation whose range includes \mathbb{Q}^+ .
 - 3. If $\langle A, \succeq \rangle \in \mathscr{A}$, then it is part of a conjoint structure in the sense that either:
 - (i) $A = A_1 \times A_2$, $\langle A_1 \times A_2, \succeq \rangle$ is a symmetric conjoint structure with a multiplicative representation, and $\langle A_i, \succeq_i \rangle$ are in \mathscr{A} ; or
 - (ii) there is a symmetric conjoint structure $\langle A'_1 \times A'_2, \succeq' \rangle \in \mathscr{A}$ with a multiplicative representation such that $A'_1 = A$, $\succeq'_1 = \succeq$, and $\langle A'_2, \succeq'_2 \rangle \in \mathscr{A}$.

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Embedding into a Structure of Physical Quantities

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    Axiamatika of and δ as follows:
    (1) (β<sub>k</sub> × β<sub>k</sub>) ≥ d<sup>k</sup>, then either
    (1) (β<sub>k</sub> × β<sub>k</sub>) ≥ d<sup>k</sup>, then either
    (1) there exist η on β<sub>k</sub> = 1.2, such that (β<sub>k</sub> ≥<sub>k</sub>, η<sub>k</sub>) are both in δ and a line of element is on β<sub>k</sub> × β<sub>k</sub> and line titler i = 1 = 0.2, η<sub>k</sub> on A such that (β<sub>k</sub> × β<sub>k</sub>) ≥ d<sup>k</sup>, on (β<sub>k</sub> > 0) and (β<sub>k</sub> > 1, η<sub>k</sub>) which is d<sup>k</sup> and an either i = 1 = 0.2, η<sub>k</sub> on A such that (β<sub>k</sub> × β<sub>k</sub>) ≥ d<sup>k</sup>, on (β<sub>k</sub> > 0) and (β<sub>k</sub> > 1, η<sub>k</sub>) = 0, in (1.2) there i = 0, n = 0, in (β<sub>k</sub> = 0, η<sub>k</sub>) = 0, in (1.2) there i = 0, in (1.2) there i = 0, n = 0, in (1.2) there i = 0, n = 0, in (1.2) there i = 0, n = 0, in (1.2) there i = 0, n = 0, in (1.2) there i = 0, n = 0, in (1.2) there i = 0, n = 0, in (1.2) there i = 0, n = 0, in (1.2) there i = 0, n = 0, in (1.2) there i = 0, n = 0, n = 0, in (1.2) there i = 0, n = 0, n = 0, n = 0, in (1.2) there i = 0, n < 0, n </li>
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Embedding into a Structure of Physical Quantities

- Axiomatize \mathscr{A} and \mathscr{E} as follows:
 - 4. If $\langle A_1 \times A_2, \succeq \rangle \in \mathscr{A}$, then either
 - (i) there exist \circ_i on A_i , i = 1, 2, such that $\langle A_i, \succeq_i, \circ_i \rangle$ are both in \mathscr{E} and a law of exchange holds; or
 - (ii) there exist \circ on $A_1 \times A_2$ and for either i = 1 or 2, \circ_i on A_i such that $\langle A_1 \times A_2, \succeq, \circ \rangle$ and $\langle A_i, \succeq_i, \circ_i \rangle$ are both in \mathscr{E} and a law of similitude holds.
 - 5. Suppose laws of similitude hold both for $\langle A_1 \times A_2, \succeq, \circ, \circ_1 \rangle$ and $\langle A_1 \times A_2, \succeq', \circ', \circ'_1 \rangle$. If $\succeq'_i = \succeq_i$ or \preccurlyeq_i , i = 1, 2, then $\succeq' = \succeq$ and $\circ' = \circ$.
Dimensional Analysis

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Embedding into a Structure of Physical Quantities

Embedding into a Structure of Physical Quantities

Embedding into a Structure of Physical Quantities

Suppose assumptions 1-5 hold. Then there exists a subset *B* of *B* that is maximal with respect to the properties: (i) not both an attribute and its converse are in *S* (ii) no law of exchange or similitude holds with all three attributes in R. Further, if ϕ_1, \dots, ϕ_n are extensive representations of the *n* attributes in \mathscr{B} and if ψ is a representation of an attribute in \mathscr{A} . then there exist unique real $\alpha > 0$ and unique rational ρ_1 such $\psi = \prod_{i=1}^{n} \phi_{i}^{\mu_{i}}$

Embedding into a Structure of Physical Quantities

Theorem 10

Suppose assumptions 1-5 hold. Then there exists a subset \mathscr{B} of \mathscr{E} that is maximal with respect to the properties:

- (i) not both an attribute and its converse are in \mathscr{B}
- (ii) no law of exchange or similitude holds with all three attributes in \mathcal{B} .

Further, if ϕ_1, \ldots, ϕ_n are extensive representations of the *n* attributes in \mathscr{B} and if ψ is a representation of an attribute in \mathscr{A} , then there exist unique real $\alpha > 0$ and unique rational ρ_i such that

$$\psi = \prod_{i=1}^{n} \phi_i^{\rho_i}$$

└── Dimensional Analysis

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-Embedding into a Structure of Physical Quantities

-Embedding into a Structure of Physical Quantities

Embedding into a Structure of Physical Quantities Suppose that the assumptions of Theorem 10 hold and let *3* and $A = \left\{ \alpha \prod_{i=1}^{n} \phi_{i}^{\rho_{i}} | \alpha \in \mathbb{R}, \rho_{i} \in \mathbb{Q} \right\}, A^{+} = \left\{ \alpha \prod_{i=1}^{n} \phi_{i}^{\rho_{i}} | \alpha \in \mathbb{R}^{+}, \rho_{i} \in \mathbb{Q} \right\}$ and let + denote pointwise multiplication of functions from A. (i) (A, A⁺, *) is a structure of physical quantities (ii) {φ₁,..., φ₈} is a basis of the structure (iii) if ψ is a representation of an attribute in α' , the $\psi \in A$

What the theorem shows is that the axioms of extensive and conjoint measurement plus some assumptions about the occurrence of two types of trinary laws are adequate to construct a structure of physical quantities that satisfies the usual axions. Moreover, it shows that there is a basis composed entirely of extensive representations.

Embedding into a Structure of Physical Quantities

Theorem 11

Suppose that the assumptions of Theorem 10 hold and let ${\cal B}$ and ϕ_i be defined as there. Let

$$\mathsf{A} = \left\{ \alpha \prod_{i=1}^{n} \phi_{i}^{\rho_{i}} | \alpha \in \mathbb{R}, \ \rho_{i} \in \mathbb{Q} \right\}, \mathsf{A}^{+} = \left\{ \alpha \prod_{i=1}^{n} \phi_{i}^{\rho_{i}} | \alpha \in \mathbb{R}^{+}, \ \rho_{i} \in \mathbb{Q} \right\}$$

and let * denote pointwise multiplication of functions from A. Then

(i) $\langle A, A^+, * \rangle$ is a structure of physical quantities

(ii) $\{\phi_1,\ldots,\phi_n\}$ is a basis of the structure

(iii) if ψ is a representation of an attribute in \mathscr{A} , the $\psi \in A^+$.



Outline Macaurement Inequalities Dimensional Analogies Why are Numerical Laws Dimensionality Invariant?

(transition)

Outline

Measurement Inequalities

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Dimensional Analysis

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Why are Numerical Laws Dimensionally Invariant?

-Why are Laws Dimensionally Invariant?

Why are Laws Dimensionally Invariant?

 Given the spirit of the protocol sections, we would like to foundation a spiritual quadration definition of a physical have the spiritual have a spiritual quadratic definition of the bit of dimensional physical. However, the authors users would be arrive at a find sector 4 characterization.
There have been there classics of attempts to account for dimensional invariance.
"Description identication".
"Description identication".

Why are Laws Dimensionally Invariant?

- Given the spirit of the previous sections, we would like to formulate a general qualitative definition of a physical law, using only orderings and concatenations, and then prove that it is dimensionally invariant. However, the authors were unable to arrive at or find such a characterization.
- There have been three classes of attempts to account for dimensional invariance:
 - 1. "It couldn't be otherwise"
 - 2. "Descriptive/deductive"
 - 3. "Physical similarity"

Dimensional Analysis

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- -Why are Numerical Laws Dimensionally Invariant?
 - -Why are Laws Dimensionally Invariant?

Why are Laws Dimensionally Invariant?

Argument Scher

Choice of units is entirely arbitrary

 Any assertion that describes physical phenomena cannot depend on something entirely arbitrary.
Therefore, descriptions of physical phenomena must be dimensionally luvariant.

We suspect that many who hold this view are simply saying...that if we know how to formulate what we mean by a qualitative physical law, then we would first, as a purely logical consequence of our measurement assumptions, that the ramerical representation of the law would be dimensionality invariant.⁴⁷ [505]

Why are Laws Dimensionally Invariant?

It couldn't be otherwise

Argument Scheme

- 1. Choice of units is entirely arbitrary.
- 2. Any assertion that describes physical phenomena cannot depend on something entirely arbitrary.
- 3. Therefore, descriptions of physical phenomena must be dimensionally invariant.

"We suspect that many who hold this view are simply saying... that if we knew how to formulate what we mean by a qualitative physical law, then we would find, as a purely logical consequence of our measurement assumptions, that the numerical representation of the law would be dimensionally invariant." (505) ^{62 of 70}

Dimensional Analysis

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-Why are Numerical Laws Dimensionally Invariant?

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Why are Laws Dimensionally Invariant?

Argument Schem

 Fundamental physical laws are, as a matter of fact, dimensionally invariant.
All laws that derive from dimensionally invariant laws are

dimensionally invariant. Therefore, all physical laws are dimensionally invariant.

 Only accounts for derived laws, doesn't justify the use of dimensional analysis to obtain new results.
Derived laws depend not only on the fundamental laws but also on boundary conditions (not a problem if the boandary conditions are dimensionally invariant). Why are Laws Dimensionally Invariant? Descriptive/deductive

Argument Scheme

- 1. Fundamental physical laws are, as a matter of fact, dimensionally invariant.
- 2. All laws that derive from dimensionally invariant laws are dimensionally invariant.
- 3. Therefore, all physical laws are dimensionally invariant.
- Only accounts for derived laws, doesn't justify the use of dimensional analysis to obtain new results.
- Derived laws depend not only on the fundamental laws but also on boundary conditions (not a problem if the boundary conditions are dimensionally invariant).

Dimensional Analysis

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Why are Numerical Laws Dimensionally Invariant?

Why are Laws Dimensionally Invariant? Physical similarity

• Consider a system of positive dimensions $P_1, ..., P_r$. • Let (A, A^*, \circ) be the hat-d-dimensional structure of physical quantities on the dimensions $P_1, ..., P_r$. Items $p \in A$ are just elements $p \in \prod_{i=1}^{r} P_i$ (where formal negative elements have been appended). • Write $\mathcal{P} = \prod_{i=1}^{r} P_i$. The set of all possible configurations of a

Write P = ∏ P_i. The set of all possible configurations of a system is a set S ⊆ P.
Define an equivalence relation on sets S, S' ⊆ P by calling S

 Dome an equivalence relation on sets 3, 3 (2.5) (g) cating 3 and 5' similarity of k is the image of S under a similarity on A
Let *J* designate an equivalence class under this relation, called a "family of similar sets".

- 1. Recall that a positive dimension is an equivalence class $[a^+]$. 2. Example: Springs – P_1 , P_2 represent force and length.
- 3. The set *S* is the set of all physically consistent force-length combinations for springs of a fixed spring-constant value.

Why are Laws Dimensionally Invariant? Physical similarity

- Consider a system of positive dimensions P_1, \ldots, P_r .
- Let $\langle A, A^+, * \rangle$ be the finite-dimensional structure of physical quantities on the dimensions P_1, \ldots, P_r . Items $p \in A$ are just elements $p \in \prod_{i=1}^r P_i$ (where formal negative elements have been appended).
- Write $\mathscr{P} = \prod_{i=1}^{r} P_i$. The set of all possible configurations of a system is a set $S \subseteq \mathscr{P}$.
- Define an equivalence relation on sets $S, S' \subseteq \mathscr{P}$ by calling S and S' similar iff S' is the image of S under a similarity on A.
- Let *I* designate an equivalence class under this relation, called a "family of similar sets".

Dimensional Analysis

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-Why are Numerical Laws Dimensionally Invariant?

Why are Laws Dimensionally Invariant? Physical similarity

 The behavior of (at least some) physical systems can be described as subsets of some *P*.
Two physical systems "of the same type" can be described as

 Iwe physical systems "of the same type" can be described as subsets of the same *P*, and these subsets are similar.
If a subset of *P* describes the behavior of a physical system

and if another subset is similar to it then there is a physical system of the same type whose behavior is described by the second subset.

1. Similarities carry the set of consistent values for one spring-constant into those for another spring-constant.

Why are Laws Dimensionally Invariant? Physical similarity

- The behavior of (at least some) physical systems can be described as subsets of some *P*.
- Two physical systems "of the same type" can be described as subsets of the same *P*, and these subsets are similar.
- If a subset of *P* describes the behavior of a physical system and if another subset is similar to it, then there is a physical system of the same type whose behavior is described by the second subset.

Dimensional Analysis

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Why are Numerical Laws Dimensionally Invariant?

-Why are Laws Dimensionally Invariant?

Why are Laws Dimensionally Invariant?

Why are Laws Dimensionally Invariant? Physical similarity

- Physical theory also associates a unique set of dimensional constants to each system in a family of similar systems.
- Some additional positive dimensions Q_1, \ldots, Q_t of $\langle A, A^+, * \rangle$ are singled out. Let $\mathscr{Q} = \prod_{j=1}^t Q_j$.
- We want a function g: I → 2 associating a t-tuple of dimensional constants with each system S ∈ I in a consistent way. In particular, we need g ∘ φ = φ ∘ g for all similarities φ on A. Such a g is called a system measure of I.

└── Dimensional Analysis

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Why are Numerical Laws Dimensionally Invariant?

-Why are Laws Dimensionally Invariant?

Why are Laws Dimensionally Invariant?

Law Satisfaction

Suppose \mathcal{F} is a family of similar systems, g is a system measure from \mathcal{F} into \mathcal{G} , and $f: \mathcal{P} \times \mathcal{Q} \to \mathbb{R}$. We say that \mathcal{F} satisfies the faw (f,g) iffs for all $g \in \mathcal{G}$ and $alg \in \mathcal{G}$, we have f(g,g) = 0 iff there is some $S \in \mathcal{F}$ such that $p \in S$ and g(S) = q.

mensional Invariance

A law (f,g) as above is said to be dimensionally invariant if f is a dimensionally invariant function, i.e., f(p,q)=0 iff $f(\phi(p),\phi(q))=0$ for all similarities ϕ on A.

Why are Laws Dimensionally Invariant? Physical similarity

Law Satisfaction

Suppose \mathscr{I} is a family of similar systems, g is a system measure from \mathscr{I} into \mathscr{Q} , and $f: \mathscr{P} \times \mathscr{Q} \to \mathbb{R}$. We say that \mathscr{I} satisfies the law (f,g) iff, for all $p \in \mathscr{P}$ and all $q \in \mathscr{Q}$, we have f(p,q) = 0 iff there is some $S \in \mathscr{I}$ such that $p \in S$ and g(S) = q.

Dimensional Invariance

A law (f,g) as above is said to be *dimensionally invariant* if f is a dimensionally invariant function, i.e., f(p,q) = 0 iff $f(\phi(p), \phi(q)) = 0$ for all similarities ϕ on A.

Dimensional Analysis

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-Why are Numerical Laws Dimensionally Invariant?

Why are Laws Dimensionally Invariant?

Why are Laws Dimensionally Invariant?

■ For f: 𝔅 𝔅 𝔅 𝔅 𝔅, for each q ∈ 𝔅 we can define a set S_q = {p|f(p, q) = 0}.
■ Denote by 𝔅_f the set of all nonempty S_q.

= \mathscr{I}_{ℓ} is a family of similar systems if ℓ is dimensionally invariant.

Stability Group We define the stability group of \mathscr{I} to be $SG(\mathscr{I}) = \{\psi | \psi(S) = S \forall S \in \mathscr{I}\}, \text{ where the } \psi \text{ are similarities on } \}$

Similarly, the stability group of \mathscr{D} is $SG(\mathscr{D}) = \{\psi | \psi(q) = q \ \forall q \in \mathscr{D}\},$ where the ψ are similarities on A Why are Laws Dimensionally Invariant? Physical similarity

- For $f: \mathscr{P} \times \mathscr{Q} \to \mathbb{R}$, for each $q \in \mathscr{Q}$ we can define a set $S_q = \{p | f(p,q) = 0\}.$
- Denote by \mathscr{I}_f the set of all nonempty S_q .
- \mathcal{I}_f is a family of similar systems if f is dimensionally invariant.

Stability Group

We define the *stability group* of \mathscr{I} to be $SG(\mathscr{I}) = \{\psi | \psi(S) = S \ \forall S \in \mathscr{I}\}$, where the ψ are similarities on A.

Similarly, the stability group of \mathscr{Q} is $SG(\mathscr{Q}) = \{\psi | \psi(q) = q \ \forall q \in \mathscr{Q}\}$, where the ψ are similarities on A.

Dimensional Analysis

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Why are Numerical Laws Dimensionally Invariant?

-Why are Laws Dimensionally Invariant?

Why are Laws Dimensionally Invariant?

Theorem

$$\begin{split} & \text{Suppose } \mathcal{X} \text{ is a bundle of similar systems over } \mathcal{P}, \text{ Then TFLE: } () \\ & \text{ (i) There exists a space measure <math display="inline">g$$
 from $1 \text{ visto } \mathcal{I}. \end{cases} \\ & \text{(i) There exists a space time for a space of the star is a statistic stree of the star is a star$

Why are Laws Dimensionally Invariant? Physical similarity

Theorem 12

Suppose ${\mathscr I}$ is a family of similar systems over ${\mathscr P}.$ Then TFAE:

(i) There exists a system measure g from \mathscr{I} into \mathscr{Q} .

- (ii) There exists a function f from $\mathscr{P} \times \mathscr{Q}$ into \mathbb{R} and a function g from \mathscr{I} into \mathscr{Q} such that \mathscr{I} satisfies the dimensionally invariant law (f,g).
- (iii) $SG(\mathscr{I}) \subseteq SG(\mathscr{Q})$

Assuming the above, then TFAE:

(iv) The system measure g is injective.

$$(\mathsf{v}) \ \mathscr{I}_{f} = \mathscr{I}.$$

(vi) $SG(\mathscr{I}) \supseteq SG(\mathscr{Q})$.

Dimensional Analysis

2011-04-25

Why are Numerical Laws Dimensionally Invariant?

-Why are Laws Dimensionally Invariant?

Why are Laws Dimensionally Invariant?

Theorem 12

Uniqueness: Suppose g is a system measure into \mathcal{D} . Then g' is a system measure into \mathcal{D} fifthere is a similarity ϕ on A such that $g' \to \phi_S$ if $g' \to \phi_S$ and $f'(\rho, q) = f'(\phi, q)$, then \mathcal{J} satisfies the dimensionally invariant law (f, g) if \mathcal{J} axisfies the dimensionally invariant law (f, g).

Why are Laws Dimensionally Invariant? Physical similarity

Theorem 12

Uniqueness: Suppose g is a system measure into \mathscr{Q} . Then g' is a system measure into \mathscr{Q} iff there is a similarity ϕ on A such that $g' = \phi \circ g$. If $g' = \phi \circ g$ and $f'(p,q) = f(\phi(p),q)$, then \mathscr{I} satisfies the dimensionally invariant law (f,g) iff \mathscr{I} satisfies the dimensionally invariant law (f',g').

Dimensional Analysis

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Questions

 One of the equivalent conditions was "Three exists a system measure g from J' into J?" Do all families of similar systems always have a system measure (and hence satisfy a dimensionally invariant law)?
Does an arbitrary dimensionally invariant function f always

Does an arbitrary dimensionality unvariant function I always lead to the definition of a family of similar systems S and a system measure g on S that satisfies the law (f, g)? Why are Laws Dimensionally Invariant? Physical similarity

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- Does an arbitrary dimensionally invariant function f always lead to the definition of a family of similar systems *I* and a system measure g on *I* that satisfies the law (f,g)?

Dimensional Analysis

2011-04-25

-Why are Numerical Laws Dimensionally Invariant?

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Why are Laws Dimensionally Invariant?

Answers Yes to both, for a restricted class of families of similar systems (Theorem 13, 511). The restriction is necessary because we only allowed rational powers of dimensions in structures of physical quantities.

Limitation

Doesn't account for laws involving universal constants (no distinct, realized similar systems).

Why are Laws Dimensionally Invariant? Physical similarity

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