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Measurement Inequalities and Dimensional Analysis

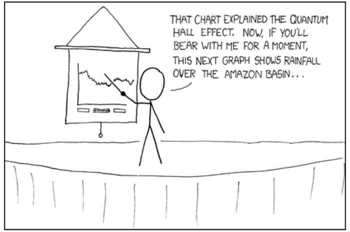
IF YOU KEEP SAYING "BEAR WITH ME FOR A MOMENT", PEOPLE TAKE A WHILE TO FIGURE OUT THAT YOU'RE JUST SHOWING THEM RANDOM SLIDES.

Gregory McWhirter

UC Irvine

Foundations of Measurement
Spring 2011

Measurement Inequalities and Dimensional Analysis



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Outline

Measurement Inequalities
Solvability
Finite Linear Structures
Polynomial Structures

Dimensional Analysis
Physical Laws
The Algebra of Physical Quantities
The Pi Theorem and Dimensional Analysis
Examples of Dimensional Analysis
Consistency of Derived Measures
Embedding into a Structure of Physical Quantities
Why are Numerical Laws Dimensionally Invariant?

1. About 1 hour on Measurement Inequalities, then a break
2. Remainder of the time on Dimensional Analysis

Outline

Outline

Measurement Inequalities
Solvability

Dimensional Analysis

(transition)

Outline

Solvability Conditions
Examples
- If $a \succ b$, then there exists $d \in A$ such that $(b, d) \in B$ and $a \succsim b \circ d$. (84)
- If $ab,\ cd \in A^*$ and $ab \succ cd$, then there exist $d',\ d'' \in A$ such that $ad',\ d'b,\ ad'',\ d''b \in A^*$ and $ad' \sim cd \sim d''b$. (147)
- Definition 6.5 (256)
- Most Closure axioms.

1. So far, we have made heavy use of various solvability axioms.

Solvability Conditions

Examples

- If $a \succ b$, then there exists $d \in A$ such that $(b, d) \in B$ and $a \succsim b \circ d$. (84)
- If $ab,\ cd \in A^*$ and $ab \succ cd$, then there exist $d',\ d'' \in A$ such that $ad',\ d'b,\ ad'',\ d''b \in A^*$ and $ad' \sim cd \sim d''b$. (147)
- Definition 6.5 (256)
- Most Closure axioms.

Failures of Solvability
- Solvability axioms are existence claims, so they are usually non-necessary.
- There are models that almost satisfy the conjoint measurement structure, for instance, but one variable is discrete and the other is not equally spaced.
- Even if solvability is a safe assumption, the shape of the data can make solving the requisite equations practically impossible.
- Even if a nontested solvability condition is true in the underlying data-generating process and if the tested necessary conditions are true in the obtained factorial data, it does not follow that the obtained data possess a representation of the kind in question. (425)

1. Solvability claims are usually non-necessary
2. Some models can get close to being, for example, a conjoint measurement structure, but they slightly miss.
3. Even if data generating processes satisfy solvability, that does not mean that the data collected also satisfy it, nor does it mean that the equations implied by the data are practically solvable.

Failures of Solvability

- Solvability axioms are existence claims, so they are usually non-necessary.
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- Even if solvability is a safe assumption, the shape of the data can make solving the requisite equations practically impossible.
- Even if a nontested solvability condition is true in the underlying data-generating process and if the tested necessary conditions are true in the obtained factorial data, it does not follow that the obtained data possess a representation of the kind in question. (425)

1. Suppose we have a $3 \times 3 \times 2$ factorial design and we get the ordinal data shown in the top two tables
2. These data do not violate the independence axioms necessary for an additive decomposition. An example of one of the checks is in the bottom left table.
3. However, these data do not satisfy double cancellation (inequalities are schematic).
4. Since solvability and independence imply double cancellation, the data generated cannot satisfy solvability.
5. So what can we do if we can't assume solvability?

Failures of Solvability

Example

Example from 425

a_3

	a_1	b_1	c_1
a_2	12	14	18
b_2	4	10	16
c_2	2	6	8

b_3

	a_1	b_1	c_1
a_2	11	13	17
b_2	3	9	15
c_2	1	5	7

	a_3	b_3
a_2	6	5
b_2	4	3
c_2	2	1

$a_1 b_2 \precsim b_1 c_2$ $\quad a_1 + b_2 \leq b_1 + c_2$

$b_1 a_2 \precsim c_1 b_2$ $\quad b_1 + a_2 \leq c_1 + b_2$

$a_1 a_2 \succ c_1 c_2$ $\quad a_1 + a_2 > c_1 + c_2$

Failures of Solvability
Example
Example from 425

(transition)

Outline

Finite Linear Structures
Additive Conjoint Models
- Suppose A_1 and A_2 are finite sets.
- Let $\succsim$ be a weak order of $A = A_1 \times A_2$.
- We want to find necessary and sufficient conditions such that $ap \succsim bq$ iff $\phi_1(a) + \phi_2(p) \geq \phi_1(b) + \phi_2(q)$.
- This is possible for any finite number of A_i given that all n-th order cancellation axioms hold.
- Furthermore, all n-th order cancellation axioms were implied by independence, double-cancellation, Archimedean-ness, and restricted solvability.

1. (read slide)
2. Since we want additive representations without solvability, we need something else to get us all of the cancellation axioms.

Finite Linear Structures

Additive Conjoint Models

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- Furthermore, all n-th order cancellation axioms were implied by independence, double-cancellation, Archimedean-ness, and restricted solvability.

Finite Linear Structures
Auxiliary Space Construction
- Let $A = A_1 \times \ldots \times A_n$.
- Suppose $|A_i| = k_i < \infty$ for all i and $A_i \cap A_j = \emptyset$ for all $i \neq j$.
- Let $\succsim$ be a reflexive binary relation on A (think a weak ordering, but it doesn't need to be transitive or connected).
- Let $k = \sum_{i=1}^{n} k_i$ be the size of $Y = \bigcup_{i=1}^{n} A_i$.
- Enumerate the elements of Y as $y_1, \ldots, y_k$.
- Define an injective mapping $v\colon A \to \mathbb{R}^k$ by $a \mapsto \overline{a} = (\overline{a}_1, \ldots, \overline{a}_k)$, where
$$\overline{a}_i = \begin{cases} 1 & \text{if } y_i \text{ is a component of } a \\ 0 & \text{otherwise} \end{cases}$$

Finite Linear Structures

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Finite Linear Structures
Auxiliary Space Construction Continued
- Let $\overline{A} = \{\overline{a} | a \in A\}$ and A^+ be the additive closure over $\overline{A}$.
- Define $\sim_I$ on A^+ by $x \sim_I y$ iff there are $\overline{a}^{(1)}, \ldots, \overline{a}^{(m)}, \overline{b}^{(1)}, \ldots, \overline{b}^{(m)} \in \overline{A}$ such that $x = \sum_{i=1}^{m} \overline{a}^{(i)}$ and $y = \sum_{i=1}^{m} \overline{b}^{(i)}$ and $\overline{a}^{(i)} \sim \overline{b}^{(i)}$ for all i.
- Define $\succ_I$ on A^+ by $x \succ_I y$ iff there are $\overline{a}^{(1)}, \ldots, \overline{a}^{(m)}, \overline{b}^{(1)}, \ldots, \overline{b}^{(m)} \in \overline{A}$ such that $x = \sum_{i=1}^{m} \overline{a}^{(i)}$ and $y = \sum_{i=1}^{m} \overline{b}^{(i)}$ and $\overline{a}^{(i)} \succsim \overline{b}^{(i)}$ for all i and for some j, $\overline{b}^{(j)} \not\succsim \overline{a}^{(j)}$.
- Define $\succsim_I$ on A^+ as $\succsim_I = \sim_I \cup \succ_I$.

Finite Linear Structures

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Finite Linear Structures
Properties of $\sim_I$, $\succ_I$, and $\succsim_I$
- The relation $\sim_I$ is reflexive and symmetric.
- The relation $\succ_I$ is not necessarily irreflexive nor asymmetric (contrary to what the usual parallel with $>$ might suggest).

Finite Linear Structures

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Finite Linear Structures

Counterexample to $\succ_I$ being necessarily asymmetric.

Example

Suppose $\succsim$ on $A_1 \times \ldots \times A_n$ violates independence. So we have the following for some a, b, a', b':

$$a = a_1 \cdots a_i \cdots a_n \succ b_1 \cdots a_i \cdots b_n = b'$$
$$b = b_1 \cdots b_i \cdots b_n \succeq a_1 \cdots b_i \cdots a_n = a'$$

Finite Linear Structures
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Now, we have, WLOG:

Finite Linear Structures

Counterexample to $\succ_I$ being necessarily asymmetric.

Example

Now, we have, WLOG:

$$\overline{a} = (\underbrace{1, 0, \ldots, 0}_{A_1}, \ldots, \underbrace{1, 0, \ldots, 0}_{A_i}, \ldots, \underbrace{1, 0, \ldots, 0}_{A_n})$$

$$\overline{a'} = (\underbrace{1, 0, \ldots, 0}_{A_1}, \ldots, \underbrace{0, 1, 0, \ldots, 0}_{A_i}, \ldots, \underbrace{1, 0, \ldots, 0}_{A_n})$$

$$\overline{b} = (\underbrace{0, 1, 0, \ldots, 0}_{A_1}, \ldots, \underbrace{0, 1, 0, \ldots, 0}_{A_i}, \ldots, \underbrace{0, 1, 0, \ldots, 0}_{A_n})$$

$$\overline{b'} = (\underbrace{0, 1, 0, \ldots, 0}_{A_1}, \ldots, \underbrace{1, 0, \ldots, 0}_{A_i}, \ldots, \underbrace{0, 1, 0, \ldots, 0}_{A_n})$$

Finite Linear Structures

Counterexample to $\succ_I$ being necessarily asymmetric.

Example

Adding $\overline{a}+\overline{b}$ and $\overline{b'}+\overline{a'}$, we get:

$$\overline{a}+\overline{b} = (\underbrace{1,1,0,\ldots,0}_{A_1},\ldots,\underbrace{1,1,0,\ldots,0}_{A_i},\ldots,\underbrace{1,1,0,\ldots,0}_{A_n}) = \overline{b'}+\overline{a'}$$

We have $\overline{a}+\overline{b} \succ_I \overline{b'}+\overline{a'}$ since we have:

$$\begin{aligned} a \succ b' &\implies a \succsim b' \\ & b \succsim a' \\ a \succ b' &\implies b' \not\succsim a \end{aligned}$$

But we also have $\overline{b'}+\overline{a'} \succ_I \overline{a}+\overline{b}$ since we have equality of sum.

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We have $\overline{a}+\overline{b} \succ_I \overline{b'}+\overline{a'}$ since we have:
$a \succ b' \implies a \succsim b'$, $b \succsim a'$, $a \succ b' \implies b' \not\succsim a$
But we also have $\overline{b'}+\overline{a'} \succ_I \overline{a}+\overline{b}$ since we have equality of sum.

Finite Linear Structures
Moral of the Story
- The example before shows us that irreflexivity of $\succ_I$ implies independence of $\succsim$.
- Similarly, it can be shown that irreflexivity of $\succ_I$ implies every n-th order cancellation axiom.
- Furthermore, $\succ_I$ is irreflexive iff $\succ_I$ and $\sim_I$ are the asymmetric and symmetric parts of $\succsim_I$ respectively.
- Irreflexivity of $\succ_I$ also implies that $\succsim_I$ has no intransitive cycles, but does not imply that $\succsim_I$ is in fact transitive.

Finite Linear Structures

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Finite Linear Structures
Moral of the Story: Representation Theorem
Theorem 1
The relation $\succ_I$ is irreflexive iff there exist $\phi\colon Y \to \mathbb{R}$ and $\psi\colon A \to \mathbb{R}$ such that for all $a, b \in A$:
(i) $\psi(a) = \psi(a_1, \ldots, a_n) = \sum_{i=1}^{n} \phi(a_i)$
(ii) $a \sim b$ implies $\psi(a) = \psi(b)$
(iii) $a \succ b$ implies $\psi(a) > \psi(b)$

Finite Linear Structures

Moral of the Story: Representation Theorem

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Finite Linear Structures
Moral of the Story: Representation Theorem
Theorem 1 Proof Technique
Theorem 1 can be proved by demonstrating the existence of a vector $z \in \mathbb{R}^k$ such that:
(i) $a \sim b$ implies $z \cdot \bar{a} = z \cdot \bar{b}$
(ii) $a \succ b$ implies $z \cdot \bar{a} > z \cdot \bar{b}$
Then define $\phi(y_i) = z_i$.

Finite Linear Structures

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Finite Linear Structures
Moral of the Story: Scale Type
Theorem 2
Suppose A has an order-preserving additive representation. Then there are vectors $z^{(1)}, \ldots, z^{(m)} \in \mathbb{R}^k$ and an integer j with $0 \leq j \leq m$ such that z is an additive representation of A iff
$$z = \sum_{i=1}^{m} \alpha_i z^{(i)} + c$$
where $c = \lambda\vec{1}$, $\alpha_i \geq 0$ for $i \leq j$, and $\alpha_i > 0$ for $i > j$.
The representation is an interval scale iff $m = 1$.

Finite Linear Structures

Moral of the Story: Scale Type

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Finite Linear Structures

Before and After

Before

- Independence, double-cancellation, Archimedean-ness, and restricted solvability imply all n-th order cancellations.
- All n-th order cancellations imply additive representation.

After

- Irreflexivity of $\succ_I$ implies independence and all n-th order cancellations.
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Finite Linear Structures
Before and After
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- Independence, double-cancellation, Archimedean-ness, and restricted solvability imply all n-th order cancellations.
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After
- Irreflexivity of $\succ_I$ implies independence and all n-th order cancellations.
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Probability Structures

- We can do much the same thing for finite probability structures as well.
- Let X be a finite non-empty set, and let $\mathscr{E}$ be an algebra of sets on X, interpreted as events.
- As before, define $\overline{\mathscr{E}}$ and $\mathscr{E}^+$ and $\sim_I, \succ_I$ with $\overline{\mathscr{E}}$ and $\mathscr{E}^+$ taking the place of $\overline{A}$ and A^+ respectively.
- Let z be a representation given by Theorem 1.
- Define

  $$P(A) = \frac{z \cdot \overline{A}}{z \cdot \overline{X}}$$

  , and note that this satisfies all the requirements of probabilities. (433)
- This representation is possible iff $\succ_I$ is irreflexive (Theorem 3).

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(transition)

Outline

Polynomial Structures
- In general, factorial data and a proposed measurement model give rise to a set of polynomial inequalities.
- There is a map from $a_1 \cdots a_n \in A_1 \times \cdots \times A_n$ to a polynomial p in the unknowns corresponding to $a_1, \ldots, a_n$.
- If the proposed model is decomposable, then there is exactly one unknown for each $a_i \in A_i$, so the set of all unknowns is $Y = \bigcup_{i=1}^{n} A_i$.
- If the proposed model is not decomposable, then there may be more than one unknown for some a_i. In this case, the set of all unknowns is still designated Y.
- Define the relation $\succsim_I$ on the set of polynomials corresponding to some $a_1 \cdots a_n$ such that when p corresponds to $a_1 \cdots a_n$ and q to $b_1 \cdots b_n$, we have $p \succsim_I q$ iff $a_1 \cdots a_n \succsim b_1 \cdots b_n$.

Polynomial Structures

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Polynomial Structures
Representation Theorem: Big Picture
Theorem 4
A set of polynomial inequalities in the unknowns Y has a solution iff the corresponding relation $\succsim_I$ on $\mathbb{R}[Y]$ can be extended to a weak order $\succsim_{II}$ such that $\langle \mathbb{R}[Y], \succsim_{II} \rangle$ is an Archimedian weakly ordered ring (i.e., $\succsim_{II}$ induces an Archimedian ordered ring structure on ${}^{\mathbb{R}[Y]}\!/_{\sim_{II}}$).
- We can find necessary conditions for this extension to exist similar to the necessary and sufficient conditions from the linear case.
- However, the necessary and sufficient conditions for the extension do not imply any easily testable consequences.

Polynomial Structures

Representation Theorem: Big Picture

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Polynomial Structures
Representation Theorem: Necessary Conditions
Corollary to Theorem 5
If there exists an extension $\succsim'$ of $\succsim_I$ such that $\langle \mathbb{R}[Y], \succsim' \rangle$ is a weakly ordered ring, then $\succ^*$ is irreflexive, where $(\sim^*, \succ^*)$ is the minimal regular extension of $\succsim_I$.
Theorem 5
Any binary relation on $\mathbb{R}[Y]$ has at least one regular extension (the universal extension) and a unique minimal regular extension.
- The universal extension is $\mathbb{R}[Y] \times \mathbb{R}[Y]$

Polynomial Structures

Representation Theorem: Necessary Conditions

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Polynomial Structures
Representation Theorem: Necessary Conditions
Regular Extension
A pair of relations $(\sim_{II}, \succ_{II})$ is called a *regular extension* of $\succsim_I$ iff
(a) $p \sim_{II} q$ whenever one of the following holds:
(i) Extension: $p \sim_I q$
(ii) Polynomial Identity: $p = q$
(iii) Closure: There are p_1, p_2, q_1, q_2 with $p_1 \sim_{II} q_1$, $p_2 \sim_{II} q_2$ such that either $p = p_1 + p_2$, $q = q_1 + q_2$ or $p = p_1 p_2$, $q = q_1 q_2$.
(b) $p \succ_{II} q$ whenever one of the following holds:
(i) Extension: $p \succ_I q$
(ii) Additive Closure: There are p_1, p_2, q_1, q_2 with $p_1 \succ_{II} q_1$, $p_2 \sim_{II} q_2$ such that $p = p_1 + p_2$ and $q = q_1 + q_2$.
(iii) Multiplicative Closure: There are p_1, q_1, r with either $p_1 \succ_{II} q_1$, $r \succ_{II} 0$ or $q_1 \succ_{II} p_1$, $0 \succ_{II} r$ such that $p = p_1 r$, $q = q_1 r$.

Polynomial Structures

Representation Theorem: Necessary Conditions

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- (i) Extension: $p \succ_I q$
- (ii) Additive Closure: There are p_1, p_2, q_1, q_2 with $p_1 \succ_{II} q_1$, $p_2 \sim_{II} q_2$ such that $p = p_1 + p_2$ and $q = q_1 + q_2$.
- (iii) Multiplicative Closure: There are p_1, q_1, r with either $p_1 \succ_{II} q_1$, $r \succ_{II} 0$ or $q_1 \succ_{II} p_1$, $0 \succ_{II} r$ such that $p = p_1 r$, $q = q_1 r$.

Polynomial Structures
Necessary and Sufficient Conditions
Theorem 6
A set of polynomial inequalities in the unknowns of Y has a solution iff the corresponding relation $\succsim_I$ on $\mathbb{R}[Y]$ has a regular extension $(\sim_{II}, \succ_{II})$ such that $\succsim_{II}$ is Archimedean and $\succ_{II}$ is non-universal.
Conjecture
There exists an extension $\succsim'$ of $\succsim_I$ such that $\langle \mathbb{R}[Y], \succsim' \rangle$ is a weakly ordered ring iff $\succ^*$ is irreflexive, where $(\sim^*, \succ^*)$ is the minimal regular extension of $\succsim_I$.

1. (read slide)
2. There is a paper from the JOURNAL OF MATHEMATICAL PSYCHOLOGY 12, 99-113 (1975) by Marcel Richter that may or may not actually decide this conjecture, but at any rate gives an algebraic criterion for the solvability of arbitrary finite sets of polynomial inequalities.

Polynomial Structures

Necessary and Sufficient Conditions

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Outline

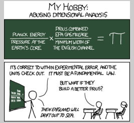
Measurement Inequalities

Dimensional Analysis
Physical Laws
The Algebra of Physical Quantities
The Pi Theorem and Dimensional Analysis
Examples of Dimensional Analysis
Consistency of Derived Measures
Embedding into a Structure of Physical Quantities
Why are Numerical Laws Dimensionally Invariant?

(transition)

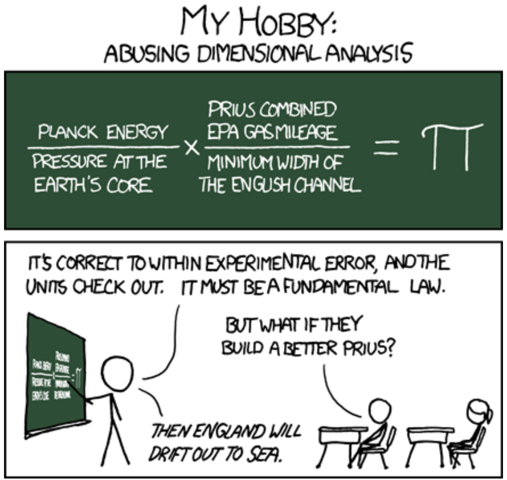
Outline

The Big Picture



xkcd.com

The Big Picture



xkcd.com

Outline

Measurement Inequalities

Dimensional Analysis
Physical Laws

(transition)

Outline

Physical Laws
Examples
- $F = ma$
- $p = mv$
- $E_k = \frac{1}{2}mv^2$
- $P = IR^2$
- $F = G\frac{m_1 m_2}{r^2}$
REMEMBER: WITH GREAT POWER COMES GREAT CURRENT SQUARED TIMES RESISTANCE.
OHM NEVER FORGOT HIS DYING UNCLE'S ADVICE.
xkcd.com

Physical Laws

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Physical Laws
What's so Special?
- Several units of measurement are expressible in terms of others.
- Taking charge (Q), temperature (Θ), mass (M), length (L), time duration (T), and angle (A) as primary, all other known physical attributes are expressible as monomial combinations of these.
  - Density: dimensions of ML^{-3}
  - Frequency: dimensions of $T^{-1}A$
  - Force: dimensions of MLT^{-2}
  - Current: dimensions of QT^{-1}
  - Entropy: dimensions of $\Theta^{-1}ML^2T^{-2}$
- In fact, all the meaningful monomial combinations known are relatively simple: $Q^\chi \Theta^\theta M^\mu L^\lambda T^\tau A^\alpha$ where $\chi, \theta, \mu, \lambda, \tau, \alpha$ are all small integers (between -4 and 4).

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Physical Laws
What's so Special?
- Furthermore, there are some "dimensional constants" that relate various measurements. Some are system-dependent, others are truly constant for a fixed system of units:
  - System-dependent gravitational constant g (e.g., approx. $9.8m/s^2$ for Earth)
  - Velocity of light c, electron charge e, gas constant R, Planck's constant h, Avogadro's constant N_A
- Certain measures such as momentum and kinetic energy are useful in many laws, but no laws seem to play a role in defining them. They are like the density of objects, not the density of materials (density independent of volume).
- Furthermore, most quantities of the form $m^i v^j$ aren't terribly important.

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Physical Laws
The Big Questions
- So what role are laws playing?
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Outline

Measurement Inequalities

Dimensional Analysis
The Algebra of Physical Quantities

(transition)

Outline

The Algebra of Physical Quantities
General Requirements
- Quantities with the same (extensively measurable) units combine additively.
- Quantities with different dimensions combine multiplicatively.
- The multiplicative structure resembles a finite-dimensional vectors space over $\mathbb{Q}$.
- The existence of basic dimensions is analogous to the existence of a finite basis of that vector space.
- Numerical physical laws are formulated in terms of a very special class of functions on the space.

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The Algebra of Physical Quantities
Axiom System
Structure of Physical Quantities
Suppose $A \hookleftarrow \mathbb{R}$ is a nonempty set, $A^+ \subseteq A$ nonempty, and $*\colon A \times A \to A$. Then $\langle A, A^+, * \rangle$ is a *structure of physical quantities* iff $\langle A \setminus \{0\}, * \rangle$ is an abelian group extension of $\langle \mathbb{R} \setminus \{0\}, \times \rangle$ and:
1. $*$ is associative and commutative.
2. $\mathbb{R} \cap A^+ = \mathbb{R}^+$.
3. $1 * a = a$ and $0 * a = 0$.
4. If $a \neq 0$, then exactly one of a and $-1 * a$ is in A^+.
5. If $x, y \in A^+$, then $x * y \in A^+$.
6. If $n \in \mathbb{Z}$, $n \neq 0$ and $x \in A^+$, there exists a unique $x^{1/n} \in A^+$ such that $(x^{1/n})^n = x$.

The Algebra of Physical Quantities

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The Algebra of Physical Quantities
Example
- Let $A = (\mathbb{R} \setminus \{0\} \times \mathbb{Q} \times \mathbb{Q}) \cup \{z\}$.
  - The element (α, q, r) can represent a quantity α with units $L^q M^r$.
  - The element z represents 0.
- Let $*$ on A be defined for non-zero operands as $(\alpha, q, r) * (\alpha', q', r') = (\alpha\alpha', q + q', r + r')$. Let $z * a = a * z = z$ for all $a \in A$.
- $\mathbb{R} \hookrightarrow A$ by $\alpha \mapsto \begin{cases} (\alpha, 0, 0) & \alpha \neq 0 \\ z & \alpha = 0 \end{cases}$
- $A^+ = \{(\alpha, q, r) | \alpha \in \mathbb{R}^+\}$
- $(\alpha, q, r)^{-1} = (\alpha^{-1}, -q, -r)$
- $(\alpha, q, r)^{1/n} = (\alpha^{1/n}, q/n, r/n)$ for $(\alpha, q, r) \in A^+$, $n \in \mathbb{Z}$.

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1. Recall that Theorem 1 was one of our desiderata.

The Algebra of Physical Quantities

Dimension Space

- Let $\langle A, A^+, * \rangle$ be a structure of physical quantities.
- For $a \neq z \in A$, define $[a] = \{\alpha * a | \alpha \in \mathbb{R}\}$, $[a^+] = [a] \cap A^+$.
- The set $[A] = \{[a] | a \in A\}$ is a set of equivalence classes over A, and each equivalence class can be thought of as a dimension.
- There are well-defined operations $[a] * [b] = [a * b]$ and $[x]^\rho = [x^\rho] = [(x^1/j)^i]$ for $x \in A^+$, $\rho = i/j$, $i, j \in \mathbb{Z}$.

Theorem 1

Suppose that $\langle A, A^+, * \rangle$ is a structure of physical quantities. Then the set $[A]$ under $*$ and powers as defined above is a multiplicative vector space over $\mathbb{Q}$ where $[1] = \mathbb{R}$ is the identity element and $[a]^{-1} = [a^{-1}]$ is the inverse of $[a]$.

The Algebra of Physical Quantities
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1. These are two more of our desiderata.
2. The formal addition agrees with the extensive concatenation operation if the dimension is extensively measurable.

The Algebra of Physical Quantities

More Desiderata

Theorem 2

Suppose that $\langle A, A^+, * \rangle$ is a structure of physical quantities. Then the elements $a_1, \ldots, a_n \in A^+$ span A iff for every $a \in A$ there exist $\alpha \in \mathbb{R}$ and $\rho_1, \ldots, \rho_n \in \mathbb{Q}$ such that $a = \alpha * a_1^{\rho_1} * \cdots * a_n^{\rho_n}$. They are independent iff $a_1^{\gamma_1} * \cdots * a_n^{\gamma_n} \in \mathbb{R}$ implies that $\gamma_i = 0$ for all i. If they are independent, they are a basis for $[A]$ and the ρ_i depend only on $[a]$.

- The dimensions that are elements of a basis for $[A]$ can be thought of as basic/fundamental dimensions.
- We can also introduce a formal addition within a dimension:
  - Suppose $a, b \in [c]$ where $a = \alpha * c$, $b = \beta * c$, and $\alpha, \beta \in \mathbb{R}$. Then define $a \oplus b = (\alpha + \beta) * c$

The Algebra of Physical Quantities
More Desiderata
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The Algebra of Physical Quantities
Functional Form Restrictions
- Let $\langle A, A^+, * \rangle$ be a structure of physical quantities.
- Let $P = [a] \cap A^+$ be a typical positive dimension.
- Physical laws have the following form:
  - A function $f : P_1 \times \cdots \times P_s \to \mathbb{R}$, where $s \geq 2$.
  - A condition $f(x_1, \ldots, x_s) = 0$ on the physically realizable values of $x_i \in P_i$.
- The functional form of laws are (usually) restricted to be *dimensionally invariant* (*homogeneous*). This means that the function should be invariant under changes of units between coherent systems.

1. In the condition on realizable values, the x_i are usually treated as real numbers, but in fact, they involve the specification of both the dimension and the unit of measurement of which the numerical dimensionless ration x_i is given.

The Algebra of Physical Quantities

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(transition)

Outline

The Pi Theorem and Dimensional Analysis

Similarities

Similarity

Suppose that $\langle A, A^+, * \rangle$ is a structure of physical quantities. A function $\phi : A \to A$ is a *similarity* iff it is an automorphism of A that preserves dimensions, maps A^+ into itself, and fixes $\alpha \in \mathbb{R}$.

Theorem 3

Suppose that a structure of physical quantities $\langle A, A^+, * \rangle$ is of finite dimension and that $\{a_1, \ldots, a_n\}$ is a basis. If ϕ is a similarity on A, then there are numbers $\phi_i > 0$ such that $\phi(a_i) = \phi_i * a$ and so $\phi(a) = (\phi_1^{\rho_1} \cdots \phi_n^{\rho_n}) * a$, where $a = \alpha * a_1^{\rho_1} * \cdots * a_n^{\rho_n}$. Conversely, for any $\phi_i > 0$, the function $\phi(a) = (\phi_1^{\rho_1} \cdots \phi_n^{\rho_n}) * a$ is a similarity.

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The Pi Theorem and Dimensional Analysis
Dimensional Invariance
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Suppose that $\langle A, A^{+}, * \rangle$ is a structure of physical quantities and that P_i are positive dimensions. A function $f\colon \prod_{i=1}^{n} P_i \to \mathbb{R}$ is *dimensionally invariant* iff for all similarities ϕ on A, $f(x_1, \ldots, x_n) = 0$ iff $f(\phi(x_1), \ldots, \phi(x_n)) = 0$.

The Pi Theorem and Dimensional Analysis

Dimensional Invariance

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The Pi Theorem and Dimensional Analysis
The Pi Theorem
Theorem 4
Suppose that $\langle A, A^+, * \rangle$ is a finite-dimensional structure of physical quantities, that P_i, $i = 1, \ldots, s$ are positive dimensions of the structure that are indexed so that the first $r < s$ form a maximal independent subset of the subspace spanned by all s of them, and that $f\colon \prod_{i=1}^{s} P_i \to \mathbb{R}$ is a dimensionally invariant function. Then there exist a function $F\colon \mathbb{R}^{s-r} \to \mathbb{R}$ and $\rho_{ij} \in \mathbb{Q}$ for $i = r+1, \ldots, s$, $j = 1, \ldots, r$ such that for all $x_i \in P_i$, $\pi_{i-r} = x_i * x_1^{-\rho_{i1}} * \cdots * x_r^{-\rho_{ir}}$, for $i = r+1, \ldots, s$, are real numbers (dimensionless), and $f(x_1, \ldots, x_s) = 0$ iff $F(\pi_1, \ldots, \pi_{s-r}) = 0$. Conversely, any function of the π's as above is dimensionally invariant.

1. To understand this, it helps to have a bit of an "example".

The Pi Theorem and Dimensional Analysis

The Pi Theorem

Theorem 4

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The Pi Theorem and Dimensional Analysis
The Pi Theorem: "Example"
- A physical law usually represents a dependent variable in terms of several independent ones: $x_s = g(x_1, \ldots, x_{s-1})$.
- Using the Pi Theorem, we can switch this to a dimensionless form: $\pi_{s-r} = G(\pi_1, \ldots, \pi_{s-r-1})$.
- We can also go backwards and express this as: $x_s = x_1^{\rho_{s1}} * \cdots * x_r^{\rho_{sr}} * G(\pi_1, \ldots, \pi_{s-r-1})$.
  - The function G gives a proportional constant relating x_s to a monomial of the independent dimensions $x_1, \ldots, x_r$.

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Outline
Measurement Inequalities
Dimensional Analysis
Examples of Dimensional Analysis

(transition)

Outline

Examples of Dimensional Analysis
A Simple Pendulum
What is the period of oscillation, t, of a simple pendulum?
The behavior of a simple pendulum has the five parameters:
- t for time
- l for the length of the pendulum
- α for the angle from vertical
- m for the mass of the pendulum
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Examples of Dimensional Analysis

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Examples of Dimensional Analysis
A Simple Pendulum
Thus, we can write down the following table:

Examples of Dimensional Analysis

A Simple Pendulum

Thus, we can write down the following table:

	Physical quantities				
Dimensions	t	l	m	g	α
L	0	1	0	1	0
M	0	0	1	0	0
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Examples of Dimensional Analysis

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Since there are three dimensions and five parameters, by the Pi Theorem, there must be 2 dimensionless parameters π_1 and π_2. Clearly one of these is $\pi_1 = \alpha$. We can use standard linear algebra to find the other.

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$$M: 0\rho_t + 0\rho_l + 1\rho_m + 0\rho_g = 0$$
$$T: 1\rho_t + 0\rho_l + 0\rho_m - 2\rho_g = 0$$

Examples of Dimensional Analysis

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$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} \rho_t \\ \rho_l \\ \rho_m \\ \rho_g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Examples of Dimensional Analysis
A Simple Pendulum
Thus, we can write down the following table:
We can choose one ρ arbitrarily. Since t is the dependent variable, it is customary to set $\rho_t = 1$.

Examples of Dimensional Analysis

A Simple Pendulum

Thus, we can write down the following table:

	Physical quantities				
Dimensions	t	l	m	g	α
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Clearly $\rho_m = 0$, and it is easy to see from the first row that $\rho_g = \frac{1}{2}$. Finally, then, $\rho_l = -\frac{1}{2}$.

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Therefore, we have $\pi_2 = t\left(\frac{g}{l}\right)^{1/2}$. Since we can write $\pi_2 = G(\pi_1)$, we then get $t = \Phi(\alpha)\left(\frac{l}{g}\right)^{1/2}$.

Examples of Dimensional Analysis

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Examples of Dimensional Analysis
Possible Errors: A Simple Pendulum
- Suppose gravitational mass and inertial mass were assumed equivalent. Then mass would have dimension $L^3 T^{-2}$. (274)
- If we walked through the simple pendulum example again, we would start with just the table:
- We would then arrive at $\pi_1 = \alpha$, $\pi_2 = l\left(\frac{g}{m}\right)^{1/2}$, and $\pi_3 = t\left(\frac{g}{l}\right)^{1/2}$.
- Therefore, we would arrive at $t = \Phi\left(l(g/m)^{1/2}, \alpha\right)\left(\frac{l}{g}\right)^{1/2}$, which is not technically wrong, but is misleading.

Examples of Dimensional Analysis

Possible Errors: A Simple Pendulum

- Suppose gravitational mass and inertial mass were assumed equivalent. Then mass would have dimension $L^3 T^{-2}$. (274)
- If we walked through the simple pendulum example again, we would start with just the table:

Dimensions	t	l	m	g	α
L	0	1	3	1	0
T	1	0	-2	-2	0

- We would then arrive at $\pi_1 = \alpha$, $\pi_2 = l\left(\frac{g}{m}\right)^{1/2}$, and $\pi_3 = t\left(\frac{g}{l}\right)^{1/2}$.
- Therefore, we would arrive at $t = \Phi\left(l(g/m)^{1/2}, \alpha\right)\left(\frac{l}{g}\right)^{1/2}$, which is not technically wrong, but is misleading.

Examples of Dimensional Analysis
Possible Errors: Ballistics

- In some situations, we may try to include more dimensions than necessary, such as with the ballistics example on 475-6. This generally leads to a more complete solution.
- Other times, redundant bases and superfluous constants may be included. This generally results in the inclusion of universal constants that can be chosen to be convenient values, reducing the solution to a non-redundant case.

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Examples of Dimensional Analysis
Obtaining Exact Solutions
- We can also use dimensional analysis to help obtain exact solutions to some partial differential equations by reducing the space of possible solutions.
- Example: Propagation of vorticity is given by
$$\frac{\partial \Omega}{\partial t} = \nu \left(\frac{\partial^2 \Omega}{\partial r^2} + \frac{1}{r} \frac{\partial \Omega}{\partial r} \right)$$
- Here, Ω is the angular velocity of a viscous fluid, r is the radial distance, t is time, and $\nu = \mu/d$ is the kinematic viscosity.
- Suppose we want to solve for $\Omega(r,t)$ subject to the initial condition that the circulation around a circle of radius R at the origin is a constant, i.e.: $\Gamma = 4\pi \int_0^R r\Omega(r,0)dr$

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Examples of Dimensional Analysis
Obtaining Exact Solutions Continued
Given this problem description, we can set up a dimensional analysis for $\Omega(\Gamma, \nu, r, t)$:
Dimensions Ω Γ ν r t; L 0 2 2 1 0; M 0 0 0 0 0; T -1 -1 -1 0 1
By the Pi Theorem, there are two dimensionless parameters, namely $\pi_1 = r^2\nu^{-1}t^{-1}$ and $\pi_2 = \Omega\nu t\Gamma^{-1}$, so we have, where $\xi = r^2/\nu t$:
$\Omega = (\Gamma/\nu t)\Phi(\xi)$

Examples of Dimensional Analysis

Obtaining Exact Solutions Continued

Given this problem description, we can set up a dimensional analysis for $\Omega(\Gamma, \nu, r, t)$:

Dimensions	Ω	Γ	ν	r	t
L	0	2	2	1	0
M	0	0	0	0	0
T	–1	–1	–1	0	1

By the Pi Theorem, there are two dimensionless parameters, namely $\pi_1 = r^2\nu^{-1}t^{-1}$ and $\pi_2 = \Omega\nu t\Gamma^{-1}$, so we have, where $\xi = r^2/\nu t$:

$$\Omega = (\Gamma/\nu t)\Phi(\xi)$$

Examples of Dimensional Analysis
Obtaining Exact Solutions Continued
Substituting $\Omega = (\Gamma/\nu t)\Phi(\xi)$ into $\frac{\partial \Omega}{\partial t} = \nu \left(\frac{\partial^2 \Omega}{\partial r^2} + \frac{1}{r}\frac{\partial \Omega}{\partial r} \right)$ and simplifying, we get:
$$\frac{d}{dt}\left[\xi \Phi(\xi) + 4\xi \frac{d\Phi(\xi)}{d\xi} \right] = 0$$
Therefore, it is clear that we must have
$$\xi \Phi(\xi) + 4\xi \frac{d\Phi(\xi)}{d\xi} = C$$
Assuming $\Phi(0)$ and $\frac{d\Phi(0)}{d\xi}$ are finite, setting $\xi = 0$ shows that $C = 0$.

Examples of Dimensional Analysis

Obtaining Exact Solutions Continued

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Examples of Dimensional Analysis

Obtaining Exact Solutions Continued

So, we can rewrite things as:

$$\frac{d\Phi(\xi)}{d\xi} = -\frac{1}{4}\Phi(\xi)$$

From this it clearly follows that we must have $\Phi(\xi) = Ae^{-\xi/4}$, for some constant A.

Substituting this back into the expression for Ω, we get $\Omega(r, t) = (\Gamma A/\nu t)e^{-r^2/4\nu t}$.

Putting that back into the initial condition to solve for $A = \frac{1}{8\pi}$, we arrive at the solution $\Omega(r, t) = (\Gamma/8\pi\nu t)e^{-r^2/4\nu t}$.

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(transition)

Outline

Consistency of Derived Measures

- The algebra of physical quantities gave us a way to describe how the ratio scale measures of various physical quantities combine, but it did not discuss the consistency of various measures obtained by other theories (e.g., extensive and conjoint measurement).
- It is generally acknowledged that there are quantities that must be measured indirectly in terms of other extensive measures, and this is only possible because various physical laws are true.
- Furthermore, we want to show how extensive and conjoint measures can be embedded as a substructure of the theory we developed earlier.

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- Furthermore, we want to show how extensive and conjoint measures can be embedded as a substructure of the theory we developed earlier.

Consistency of Derived Measures
Laws of Similitude
Consider a conjoint multiplicative scale $a = bc$ where a and b can also be extensively measured. We would like some conditions to ensure consistency in the measurements.
Law of Similitude
Suppose that $\langle A_1 \times A_2, \succsim \rangle$ is an additive conjoint structure and that $\langle A_1 \times A_2, \succsim^*, \circ \rangle$ and $\langle A_1, \succsim_1^*, \circ_1 \rangle$ are extensive structures. A (qualitative) *law of similitude with exponents m and n*, where $m, n \in \mathbb{Z}^+$ holds iff one of the following is valid for all $a \in A_1$, all $u \in A_2$, and all $i \in \mathbb{Z}^+$, where the concatenations exist:
(i) $\succsim^* = \succsim$, $\succsim_1^* = \succsim_1$ or $\succsim^* = \precsim$, $\succsim_1^* = \precsim_1$ and $i^m(a, u) \sim (i^n a, u)$
(ii) $\succsim^* = \succsim$, $\succsim_1^* = \precsim_1$ or $\succsim^* = \precsim$, $\succsim_1^* = \succsim_1$ and $(a, u) \sim i^m(i^n a, u)$

1. $\succsim^*$ is one of $\succsim$ or $\precsim$, and similarly for the second structure.

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Consistency of Derived Measures
Laws of Similitude
Theorem 5
Suppose $\langle A_1 \times A_2, \succsim \rangle$ is a flat conjoint structure that has an additive representation $\log \psi_1 + \log \psi_2$; $\langle A_1 \times A_2, \succsim^*, \circ \rangle$ and $\langle A_1, \succsim_1^*, \circ_1 \rangle$ are closed extensive structures with no essential maxima; ϕ and ϕ_1 are, respectively, additive extensive scales; and that the range of ϕ_1 includes $\mathbb{Q}^+$. If a law of similitude with exponents m and n holds, then there are constants α, γ, α_1, and γ_1 such that:
(i) $\psi_1 \psi_2 = \gamma \phi^\alpha$ and $\psi_1 = \gamma_1 \phi_1^{\alpha_1}$
(ii) $\alpha > 0$ or < 0 according as $\succsim^* = \succsim$ or $\precsim$ and $\alpha_1 > 0$ or < 0 according as $\succsim_1^* = \succsim_1$ or $\precsim_1$
(iii) $|\alpha/\alpha_1| = n/m$

1. A flat structure is one such that for every $a, b \in A_1$ there are $u, v \in A_2$ such that $(a, u) \sim (b, v)$.

Consistency of Derived Measures

Laws of Similitude

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Consistency of Derived Measures
Laws of Exchange
Now consider a conjoint multiplicative scale $a = bc$ where b and c can also be extensively measured. We would again like some conditions to ensure consistency in the measurements.
Law of Exchange
Suppose $\langle A_1 \times A_2, \succsim \rangle$ is an additive conjoint structure and $\langle A_k, \succsim_k^*, \circ_k \rangle$, k=1,2, are extensive structures. A (qualitative) *law of exchange with exponents m and n*, where $m, n \in \mathbb{Z}^+$, holds iff once of the following is valid for all $a \in A_1$, all $u \in A_2$, and all $i \in \mathbb{Z}^+$, where the concatenations exist:
(i) $\succsim_1^* = \succsim_1$, $\succsim_2^* = \succsim_2$ or $\succsim_1^* = \precsim_1$, $\succsim_2^* = \precsim_2$ and $(i^m a, u) \sim (a, i^n u)$
(ii) $\succsim_1^* = \succsim_1$, $\succsim_2^* = \precsim_2$ or $\succsim_1^* = \precsim_1$, $\succsim_2^* = \succsim_2$ and $(a, u) \sim (i^m a, i^n u)$

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Consistency of Derived Measures
Laws of Exchange
Theorem 6
Suppose $\langle A_1 \times A_2, \succsim \rangle$ is a conjoint structure that has an additive representation $\log \psi_1 + \log \psi_2$; $\langle A_k, \succsim_k^*, \circ_k \rangle$, $k = 1, 2$, are closed positive extensive structures with no essential maxima and additive representations ϕ_k. If a law of exchange with exponents m and n holds, then there are constants α_k, and γ_k, $k = 1, 2$, such that:
(i) $\psi_k = \gamma_k \phi_k^{\alpha_k}$
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Consistency of Derived Measures

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Consistency of Derived Measures
Similitude and Exchange Compatibility
- How compatible are the laws of similitude and exchange as given? Are more assumptions needed?
- Consider the case of conjoint measurement where $a = bc$ and a, b, c all have extensive measurements. Two laws of similitude and one law of exchange could possibly hold simultaneously. In this case, any two of the three possibly laws determine what the third must be for a representation of the form $\phi(a, u)^{\alpha} = \phi_1(a)^{\alpha_1}\phi_2(u)^{\alpha_2}$ to hold. With some manipulation, we can see that a compatible representation can be of the form $\phi^{nq} = \phi_1^{mq}\phi_2^{np}$ or $\phi^{nq} = \phi_1^{np}\phi_2^{mp}$.
- Similar conditions can be derived for cases larger than 2 dimensions.

1. Skipping difference structures

Consistency of Derived Measures

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- Similar conditions can be derived for cases larger than 2 dimensions.

Outline
Measurement Inequalities
Dimensional Analysis
Embedding into a Structure of Physical Quantities

(transition)

Outline

Embedding into a Structure of Physical Quantities

- Let $\mathscr{A}$ be a collection of physical attributes, represented by structures $\langle A, \succsim \rangle$, and let $\mathscr{E} \subset \mathscr{A}$ be a set of extensively measurable attributes, represented by structures $\langle A, \succsim, \circ \rangle$.

Embedding into a Structure of Physical Quantities

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Embedding into a Structure of Physical Quantities

- Axiomatize $\mathscr{A}$ and $\mathscr{E}$ as follows:
  1. The set $\mathscr{E}$ is nonempty and $\mathscr{A}$ is finite.
  2. If $\langle A, \succsim, \circ \rangle \in \mathscr{E}$, it is an extensive structure with an additive representation whose range includes $\mathbb{Q}^+$.
  3. If $\langle A, \succsim \rangle \in \mathscr{A}$, then it is part of a conjoint structure in the sense that either:
     (i) $A = A_1 \times A_2$, $\langle A_1 \times A_2, \succsim \rangle$ is a symmetric conjoint structure with a multiplicative representation, and $\langle A_i, \succsim_i \rangle$ are in $\mathscr{A}$; or
     (ii) there is a symmetric conjoint structure $\langle A_1' \times A_2', \succsim' \rangle \in \mathscr{A}$ with a multiplicative representation such that $A_1' = A$, $\succsim_1' = \succsim$, and $\langle A_2', \succsim_2' \rangle \in \mathscr{A}$.

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- Axiomatize $\mathscr{A}$ and $\mathscr{E}$ as follows:
  4. If $\langle A_1 \times A_2, \succsim \rangle \in \mathscr{A}$, then either
     (i) there exist $\circ_i$ on A_i, $i = 1, 2$, such that $\langle A_i, \succsim_i, \circ_i \rangle$ are both in $\mathscr{E}$ and a law of exchange holds; or
     (ii) there exist $\circ$ on $A_1 \times A_2$ and for either $i = 1$ or 2, $\circ_i$ on A_i such that $\langle A_1 \times A_2, \succsim, \circ \rangle$ and $\langle A_i, \succsim_i, \circ_i \rangle$ are both in $\mathscr{E}$ and a law of similitude holds.
  5. Suppose laws of similitude hold both for $\langle A_1 \times A_2, \succsim, \circ, \circ_1 \rangle$ and $\langle A_1 \times A_2, \succsim', \circ', \circ_1' \rangle$. If $\succsim_i' = \succsim_i$ or $\precsim_i$, $i = 1, 2$, then $\succsim' = \succsim$ and $\circ' = \circ$.

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Embedding into a Structure of Physical Quantities

Theorem 10

Suppose assumptions 1-5 hold. Then there exists a subset $\mathscr{B}$ of $\mathscr{E}$ that is maximal with respect to the properties:

(i) not both an attribute and its converse are in $\mathscr{B}$

(ii) no law of exchange or similitude holds with all three attributes in $\mathscr{B}$.

Further, if $\phi_1, \ldots, \phi_n$ are extensive representations of the n attributes in $\mathscr{B}$ and if ψ is a representation of an attribute in $\mathscr{A}$, then there exist unique real $\alpha > 0$ and unique rational ρ_i such that

$$\psi = \prod_{i=1}^{n} \phi_i^{\rho_i}$$

Embedding into a Structure of Physical Quantities
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Embedding into a Structure of Physical Quantities
Theorem 11
Suppose that the assumptions of Theorem 10 hold and let $\mathscr{B}$ and ϕ_i be defined as there. Let
$$A = \left\{ \alpha \prod_{i=1}^{n} \phi_i^{\rho_i} | \alpha \in \mathbb{R},\ \rho_i \in \mathbb{Q} \right\}, A^+ = \left\{ \alpha \prod_{i=1}^{n} \phi_i^{\rho_i} | \alpha \in \mathbb{R}^+,\ \rho_i \in \mathbb{Q} \right\}$$
and let $*$ denote pointwise multiplication of functions from A. Then
(i) $\langle A, A^+, * \rangle$ is a structure of physical quantities
(ii) $\{\phi_1, \ldots, \phi_n\}$ is a basis of the structure
(iii) if ψ is a representation of an attribute in $\mathscr{A}$, the $\psi \in A^+$.

What the theorem shows is that the axioms of extensive and conjoint measurement plus some assumptions about the occurrence of two types of trinary laws are adequate to construct a structure of physical quantities that satisfies the usual axions. Moreover, it shows that there is a basis composed entirely of extensive representations.

Embedding into a Structure of Physical Quantities

Theorem 11

Suppose that the assumptions of Theorem 10 hold and let $\mathscr{B}$ and ϕ_i be defined as there. Let

$$A = \left\{ \alpha \prod_{i=1}^{n} \phi_i^{\rho_i} | \alpha \in \mathbb{R},\ \rho_i \in \mathbb{Q} \right\}, A^+ = \left\{ \alpha \prod_{i=1}^{n} \phi_i^{\rho_i} | \alpha \in \mathbb{R}^+,\ \rho_i \in \mathbb{Q} \right\}$$

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Outline

Measurement Inequalities

Dimensional Analysis

Why are Numerical Laws Dimensionally Invariant?

(transition)

Outline

Why are Laws Dimensionally Invariant?
- Given the spirit of the previous sections, we would like to formulate a general qualitative definition of a physical law, using only orderings and concatenations, and then prove that it is dimensionally invariant. However, the authors were unable to arrive at or find such a characterization.
- There have been three classes of attempts to account for dimensional invariance:
1. "It couldn't be otherwise"
2. "Descriptive/deductive"
3. "Physical similarity"

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  1. "It couldn't be otherwise"
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Why are Laws Dimensionally Invariant?
It couldn't be otherwise
Argument Scheme
1. Choice of units is entirely arbitrary.
2. Any assertion that describes physical phenomena cannot depend on something entirely arbitrary.
3. Therefore, descriptions of physical phenomena must be dimensionally invariant.
"We suspect that many who hold this view are simply saying…that if we knew how to formulate what we mean by a qualitative physical law, then we would find, as a purely logical consequence of our measurement assumptions, that the numerical representation of the law would be dimensionally invariant." (505)

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Why are Laws Dimensionally Invariant?
Descriptive/deductive
Argument Scheme
1. Fundamental physical laws are, as a matter of fact, dimensionally invariant.
2. All laws that derive from dimensionally invariant laws are dimensionally invariant.
3. Therefore, all physical laws are dimensionally invariant.
- Only accounts for derived laws, doesn't justify the use of dimensional analysis to obtain new results.
- Derived laws depend not only on the fundamental laws but also on boundary conditions (not a problem if the boundary conditions are dimensionally invariant).

Why are Laws Dimensionally Invariant?

Descriptive/deductive

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Why are Laws Dimensionally Invariant?
Physical similarity
- Consider a system of positive dimensions $P_1, \ldots, P_r$.
- Let $\langle A, A^+, * \rangle$ be the finite-dimensional structure of physical quantities on the dimensions $P_1, \ldots, P_r$. Items $p \in A$ are just elements $p \in \prod_{i=1}^{r} P_i$ (where formal negative elements have been appended).
- Write $\mathscr{P} = \prod_{i=1}^{r} P_i$. The set of all possible configurations of a system is a set $S \subseteq \mathscr{P}$.
- Define an equivalence relation on sets $S, S' \subseteq \mathscr{P}$ by calling S and S' *similar* iff S' is the image of S under a similarity on A.
- Let $\mathscr{I}$ designate an equivalence class under this relation, called a "family of similar sets".

1. Recall that a positive dimension is an equivalence class $[a^+]$.
2. Example: Springs – P_1, P_2 represent force and length.
3. The set S is the set of all physically consistent force-length combinations for springs of a fixed spring-constant value.

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Why are Laws Dimensionally Invariant?
Physical similarity
- The behavior of (at least some) physical systems can be described as subsets of some $\mathscr{P}$.
- Two physical systems "of the same type" can be described as subsets of the same $\mathscr{P}$, and these subsets are similar.
- If a subset of $\mathscr{P}$ describes the behavior of a physical system and if another subset is similar to it, then there is a physical system of the same type whose behavior is described by the second subset.

1. Similarities carry the set of consistent values for one spring-constant into those for another spring-constant.

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Why are Laws Dimensionally Invariant?
Physical similarity
- Physical theory also associates a unique set of dimensional constants to each system in a family of similar systems.
- Some additional positive dimensions $Q_1, \ldots, Q_t$ of $\langle A, A^+, * \rangle$ are singled out. Let $\mathscr{Q} = \prod_{j=1}^{t} Q_j$.
- We want a function $g\colon \mathscr{I} \to \mathscr{Q}$ associating a t-tuple of dimensional constants with each system $S \in \mathscr{I}$ in a consistent way. In particular, we need $g \circ \phi = \phi \circ g$ for all similarities ϕ on A. Such a g is called a *system measure* of $\mathscr{I}$.

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Why are Laws Dimensionally Invariant?
Physical similarity
Law Satisfaction
Suppose $\mathscr{I}$ is a family of similar systems, g is a system measure from $\mathscr{I}$ into $\mathscr{Q}$, and $f : \mathscr{P} \times \mathscr{Q} \to \mathbb{R}$. We say that $\mathscr{I}$ *satisfies the law* (f, g) iff, for all $p \in \mathscr{P}$ and all $q \in \mathscr{Q}$, we have $f(p, q) = 0$ iff there is some $S \in \mathscr{I}$ such that $p \in S$ and $g(S) = q$.
Dimensional Invariance
A law (f, g) as above is said to be *dimensionally invariant* if f is a dimensionally invariant function, i.e., $f(p, q) = 0$ iff $f(\phi(p), \phi(q)) = 0$ for all similarities ϕ on A.

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Why are Laws Dimensionally Invariant?
Physical similarity
- For $f : \mathscr{P} \times \mathscr{Q} \to \mathbb{R}$, for each $q \in \mathscr{Q}$ we can define a set $S_q = \{p | f(p, q) = 0\}$.
- Denote by $\mathscr{I}_f$ the set of all nonempty S_q.
- $\mathscr{I}_f$ is a family of similar systems if f is dimensionally invariant.
Stability Group
We define the *stability group* of $\mathscr{I}$ to be $SG(\mathscr{I}) = \{\psi | \psi(S) = S \ \forall S \in \mathscr{I}\}$, where the ψ are similarities on A.
Similarly, the stability group of $\mathscr{Q}$ is $SG(\mathscr{Q}) = \{\psi | \psi(q) = q \ \forall q \in \mathscr{Q}\}$, where the ψ are similarities on A.

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Why are Laws Dimensionally Invariant?
Physical similarity
Theorem 12
Suppose $\mathscr{I}$ is a family of similar systems over $\mathscr{P}$. Then TFAE:
(i) There exists a system measure g from $\mathscr{I}$ into $\mathscr{Q}$.
(ii) There exists a function f from $\mathscr{P} \times \mathscr{Q}$ into $\mathbb{R}$ and a function g from $\mathscr{I}$ into $\mathscr{Q}$ such that $\mathscr{I}$ satisfies the dimensionally invariant law (f, g).
(iii) $SG(\mathscr{I}) \subseteq SG(\mathscr{Q})$
Assuming the above, then TFAE:
(iv) The system measure g is injective.
(v) $\mathscr{I}_f = \mathscr{I}$.
(vi) $SG(\mathscr{I}) \supseteq SG(\mathscr{Q})$.

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Why are Laws Dimensionally Invariant?
Physical similarity
Theorem 12
Uniqueness: Suppose g is a system measure into $\mathscr{Q}$. Then g' is a system measure into $\mathscr{Q}$ iff there is a similarity ϕ on A such that $g' = \phi \circ g$. If $g' = \phi \circ g$ and $f'(p, q) = f(\phi(p), q)$, then $\mathscr{I}$ satisfies the dimensionally invariant law (f, g) iff $\mathscr{I}$ satisfies the dimensionally invariant law (f', g').

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Why are Laws Dimensionally Invariant?
Physical similarity
Questions
- One of the equivalent conditions was “There exists a system measure g from $\mathscr{I}$ into $\mathscr{Q}$.” Do all families of similar systems always have a system measure (and hence satisfy a dimensionally invariant law)?
- Does an arbitrary dimensionally invariant function f always lead to the definition of a family of similar systems $\mathscr{I}$ and a system measure g on $\mathscr{I}$ that satisfies the law (f, g)?

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Why are Laws Dimensionally Invariant?

Physical similarity

Answers

Yes to both, for a restricted class of families of similar systems (Theorem 13, 511).

- The restriction is necessary because we only allowed rational powers of dimensions in structures of physical quantities.

Limitations

- Doesn't account for laws involving universal constants (no distinct, realized similar systems).

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